Practice Test 2Afor Calculus II, Math 1502, September 22, 2013

PRINT Name:

PRINT Section:

PRINT Name of TA:

This test is to be taken without calculators and notes of any sorts. The allowed time is 50 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write $1.414\ldots$. Show your work, otherwise credit cannot be given.

PRINT your name, your section number as well as the name of your TA on EVERY PAGE of this test. This is very important.

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I: (25 points) Consider the function e^{-x}.
a) Find the 4-th order Taylor polynomial $P_4(x)$ for e^{-x} and the corresponding remainder in Lagrange form.

$$e^{-x} = \sum_{k=0}^{4} (-1)^k \frac{x^k}{k!} - \frac{e^{-c} x^5}{5!}$$

where c is some number between 0 and x.
b) Using the above result compute an approximate value, call it A, for $\frac{1}{e}$
The approximate value is

$$\sum_{k=0}^{4} (-1)^k \frac{1}{k!} = \frac{1}{2} - \frac{1}{6} + \frac{1}{24} = \frac{3}{8}$$

c) Give an estimate on how accurate the value computed in b) approximates $\frac{1}{e}$, i.e., give a bound on

$$|\frac{1}{e} - A| \ ,$$

using the remainder found in a).
The remainder is negative, so we have that

$$\frac{1}{e} < \frac{3}{8}$$

On the other hand since c is between 0 and 1

$$\frac{e^{-c}}{5!} < \frac{1}{5!}$$

we find that

$$\frac{3}{8} - \frac{1}{120} < \frac{1}{e} < \frac{3}{8} \ .$$

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II: Decide whether the following series converge or diverge. State which convergence test you are going to use.

a) (8 points)

$$\sum_{k=0}^{\infty} \frac{[k!]^2}{(3k)!}$$

The ratio test yields

$$\frac{a_{k+1}}{a_k} = \frac{[(k+1)!]^2(3k)!}{[k!]^2(3k+3)!} = \frac{(k+1)^2}{(3k+3)(3k+2)(3k+1)} \to 0$$

as $k \to \infty$. Hence the series converges.

b) (8 points)

$$\sum_{k=1}^{\infty} \frac{3^{k^2}}{k!}$$

Apply again the ratio test

$$\frac{a_{k+1}}{a_k} = \frac{3^{k^2}3^{2k}3 \cdot k!}{3^{k^2}(k+1)!} = \frac{3^{2k+1}}{(k+1)} \to \infty$$

as $k \to \infty$. Hence the series diverges.

c) (9 points)

$$\sum_{k=1}^{\infty}(2+(-1)^k)(1-\frac{1}{k})^{k^2}$$

The root test yields

$$(2+(-1)^k)^{1/k}(1-\frac{1}{k})^k \to \frac{1}{e}$$

as $k \to \infty$. Since $1/e < 1$ the series converges.

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III: a) (9 points) Consider the alternating series

$$L = \sum_{k=0}^{\infty} (-1)^k 10^{-k^2}$$

Find the smallest value of N so that the N-th partial sum s_N satisfies $|L - s_N| < 10^{-16}$.

You know from alternating series theory that

$$|L - s_n| < 10^{-(n+1)^2}$$

This immediately yields that $n = 3$.

b) (8 points) Find the power series expansion for $\sinh x := \frac{1}{2}(e^x - e^{-x})$.

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

and

$$e^{-x} = \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{k!}$$

In the difference $e^x - e^{-x}$ the even terms cancel and we get

$$\sinh(x) = \sum_{m=0}^{\infty} \frac{x^{2m+1}}{(2m+1)!}$$

c) (8 points) Sum the series

$$\sum_{k=0}^{\infty} (k+2) 2^{-k}$$

Write the series as

$$\sum_{k=0}^{\infty} k2^{-k} + 2\sum_{k=0}^{\infty} 2^{-k}$$

The second term is a simple geometric series and sums up to $2 \times \frac{1}{1-1/2} = 4$. For the first terms differentiating the geometric series we get that

$$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} kx^{k-1}$$

so that

$$\frac{x}{(1-x)^2} = \sum_{k=0}^{\infty} kx^{k}$$

Hence

$$\sum_{k=0}^{\infty} k2^{-k} = \frac{1/2}{(1-1/2)^2} = 2$$

So the whole series sums up to 6.

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IV: Find the interval of convergence of the following power series. State which convergence test you are going to use for computing the radius of convergence.

a) (7 points)

$$\sum_{k=0}^{\infty} \frac{\sqrt{k!}}{k^k} x^k$$

Ratio test yields

$$\frac{a_{k+1}}{a_k} = \frac{\sqrt{(k+1)!}}{(k+1)^{k+1}} \frac{k^k}{\sqrt{k!}} = \frac{1}{\sqrt{k+1}} \left(\frac{k}{k+1}\right)^k \to 0$$

as $k \to \infty$, since the sequence

$$\left(\frac{k}{k+1}\right)^k \to \frac{1}{e}$$

as $k \to \infty$.

b) (8 points)

$$\sum_{k=1}^{\infty} (-1)^k \frac{1}{k^3} \left(\frac{x+3}{2}\right)^k$$

Ratio test yields

$$\frac{a_{k+1}}{a_k} = \left|\frac{x+3}{2}\right| < 1$$

and hence we know that the series converges absolutely for all x that satisfies the inequalities

$$-2 < x+3 < 2 \quad \text{or} \quad -5 < x < -1 \ .$$

At $x = -1$ the series is

$$\sum_{k=1}^{\infty} (-1)^k \frac{1}{k^3}$$

which converges, in fact absolutely by the p-test. At $x = -5$ the series is

$$\sum_{k=1}^{\infty} \frac{1}{k^3}$$

which converges by the p-test. Hence the interval of convergence is

$$-5 \leq x \leq -1 \ .$$

c) (10 points)

$$\sum_{k=1}^{\infty} \frac{3 + (-1)^k}{k} (x-1)^k$$

What function does this series represent in its open interval of convergence?

The interval of convergence is

$$0 < x < 2 \ .$$

The series does not converge at 0 nor does it converge at 2.

It represents the function

$$-3 \ln(2 - x) - \ln(x) \ .$$

You can see this in several ways. Differentiate the power series to get

$$\sum_{k=1}^{\infty} (3 + (-1)^k)(x-1)^{k-1} = \sum_{k=0}^{\infty} (3 - (-1)^k)(x-1)^k$$

which is a sum of two geometric series. The first one sums to

$$\frac{3}{(1 - (x-1))}$$

and the second one sums to

$$-\frac{1}{(1 + (x-1))}$$

Hence in total

$$\frac{3}{(2-x)} - \frac{1}{x}$$

Now we compute the antiderivative and get

$$-3\ln(2-x) - \ln x + C$$

and hence

$$\sum_{k=1}^{\infty} \frac{3+(-1)^k}{k}(x-1)^k = -3\ln(2-x) - \ln x + C$$

The power series vanishes at $x = 1$ and hence $C = 0$.