Practice Test 3B for Calculus II, Math 1502, October 17, 2013

PRINT Name:

PRINT Section:

PRINT Name of TA:

This test is to be taken without calculators and notes of any sorts. The allowed time is 50 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write $1.414\ldots$. Show your work, otherwise credit cannot be given.

PRINT your name, your section number as well as the name of your TA on EVERY PAGE of this test. This is very important.

PRINT Name:

PRINT Section:

PRINT Name of TA:

I: Consider the system of equations

$$x + 2y + uz = 1$$
$$-x + z = v$$
$$5x + 6y + 7z = 1$$

For which values of u and v does this system have a) no solution, b) exactly one solution, c) infinitely many solutions? Find the solution in case b) and find all the solutions in case c).

The augmented matrix is given by

$$\begin{bmatrix} 1 & 2 & u & 1 \\ -1 & 0 & 1 & v \\ 5 & 6 & 7 & 1 \end{bmatrix}$$

which is equivalent to

$$\begin{bmatrix} 1 & 2 & u & 1 \\ 0 & 2 & 1+u & 1+v \\ 0 & -4 & 7-5u & -4 \end{bmatrix}$$

which in turn is equivalent to

$$\begin{bmatrix} 1 & 2 & u & 1 \\ 0 & 2 & 1+u & 1+v \\ 0 & 0 & 9-3u & -2+2v \end{bmatrix} .$$

One can reduce this further to

$$\begin{bmatrix} 1 & 0 & -1 & -v \\ 0 & 2 & 1+u & 1+v \\ 0 & 0 & 9-3u & -2+2v \end{bmatrix} .$$

Hence, if $u = 3$ and $v \neq 1$ there is no solution. If $u \neq 3$ there is exactly one solution which is given by

$$z = -\frac{2 - 2v}{9 - 3u} \ , \ y = \frac{11 - u + 7v - 5uv}{18 - 6u} \ , \ x = \frac{-2 - 7v + 3uv}{9 - 3u}$$

For $u = 3$ and $v = 1$ there is a free variable. The reduced augmented matrix is then given by

$$\begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} .$$

Hence

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} .$$

PRINT Name:

PRINT Section:

PRINT Name of TA:

II: Let $T : R^2 \to R^2$ be the linear transformation obtained by first performing a rotation of $30°$ and then performing a reflection about the $x = y$ axis. Find the matrix associated with T.

The vector $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ first gets rotated into the vector

$$\frac{1}{2}\begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix} = \frac{\sqrt{3}}{2}\vec{e}_1 + \frac{1}{2}\vec{e}_2$$

The reflection about the $x = y$ axis maps the vector $\vec{e}_1$ into $\vec{e}_2$ and the vector $\vec{e}_2$ into $\vec{e}_1$. Hence

$$T(\vec{e}_1) = \frac{\sqrt{3}}{2}\vec{e}_2 + \frac{1}{2}\vec{e}_1 = \frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} .$$

The vector $\vec{e}_2$ gets rotated into the vector

$$\frac{1}{2}\begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix} = -\frac{1}{2}\vec{e}_1 + \frac{\sqrt{3}}{2}\vec{e}_2$$

Hence

$$T(\vec{e}_2) = -\frac{1}{2}\vec{e}_2 + \frac{\sqrt{3}}{2}\vec{e}_1 = \frac{1}{2}\begin{bmatrix} \sqrt{3} \\ -1 \end{bmatrix} .$$

Thus, the matrix associated with T is given by

$$[\,T(\vec{e}_1), T(\vec{e}_2)\,] = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

PRINT Name:

PRINT Section:

PRINT Name of TA:

III: Are the following vectors linearly independent?

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \ , \ \vec{v}_2 = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} \ , \ \vec{v}_3 = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}$$

If not, give all the possible linear combinations of the zero vector in terms of $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

Have to row reduce the augmented matrix

$$\begin{bmatrix} 1 & 3 & 3 & 0 \\ 2 & 2 & -2 & 0 \\ 1 & 4 & 5 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which leads to the reduced echelon form

$$\begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

where t is any number. Hence the vectors are not linearly independent and

$$\vec{0} = t[3\vec{v}_1 - 2\vec{v}_2 + \vec{v}_3]$$

where t is arbitrary.

PRINT Name:

PRINT Section:

PRINT Name of TA:

IV: a) What defines a linear transformation?

$$T : R^n \to R^m$$

is a linear transformation if

$$T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$$

all $\vec{u}, \vec{v} \in R^n$ and

$$T(\alpha\vec{u}) = \alpha T(\vec{u})$$

for all $\alpha \in R$ and all $\vec{u} \in R^n$.

b) Which of the following linear transformations $T : R^3 \to R^3$ is linear:

$$T_1(\vec{x}) = A\vec{x} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

where A is a 3×3 matrix. Is not since

$$T_1(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \neq T_1(\vec{x}) + T(\vec{y}) = A\vec{x} + A\vec{y} + 2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$T_2(\vec{x}) = \begin{bmatrix} |x| + z \\ z + x \\ x \end{bmatrix}$$

Is not because $|x_1 + x_2| =\neq |x_1| + |x_2|$.

$$T_3(\vec{x}) = \begin{bmatrix} x + z \\ y + x \\ x \end{bmatrix}$$

It is linear because it is of the form

$$T_3(\vec{x}) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \vec{x}$$

c) What is the matrix associated with the linear transformation

$$T(\vec{x}) = \vec{a} \times \vec{x}$$

where

$$\vec{a} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Setting

$$\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

then

$$\vec{a} \times \vec{x} = \begin{bmatrix} bz - cy \\ cx - az \\ ay - bx \end{bmatrix}$$

Hence

$$T(\vec{x}) = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix} \vec{x} \ .$$

PRINT Name:

PRINT Section:

PRINT Name of TA:

V: Consider the linear transformation $T : R^3 \to R^3$ that has the property that

$$T(\vec{e}_1 + \vec{e}_2) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \; , \; T(\vec{e}_2 + \vec{e}_3) = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \; , \; T(\vec{e}_2 - \vec{e}_3) = \begin{bmatrix} 0 \\ -6 \\ -2 \end{bmatrix}$$

Is this linear transformation onto?

We have to decide whether the vectors $T(\vec{e}_1), T(\vec{e}_2), T(\vec{e}_3)$ span R^3 or not.

Note that

$$T(2\vec{e}_2) = \begin{bmatrix} 3 \\ -6 \\ -1 \end{bmatrix}$$

and hence

$$T(\vec{e}_2) = \begin{bmatrix} \frac{3}{2} \\ -3 \\ -\frac{1}{2} \end{bmatrix}$$

Likewise

$$T(\vec{e}_3) = \begin{bmatrix} \frac{3}{2} \\ 3 \\ \frac{3}{2} \end{bmatrix}$$

so that

$$T(\vec{e}_1) = T(\vec{e}_1 + \vec{e}_2) - T(\vec{e}_2) = \begin{bmatrix} -\frac{1}{2} \\ 5 \\ \frac{3}{2} \end{bmatrix}$$

Thus, the matrix $[T(\vec{e}_1), T(\vec{e}_2), T(\vec{e}_3)]$ is given by

$$\frac{1}{2} \begin{bmatrix} -1 & 3 & 3 \\ 10 & -6 & 6 \\ 3 & -1 & 3 \end{bmatrix}$$

Let

$$\vec{b} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

be any vector in R^3. The augmented matrix is

$$\begin{bmatrix} -1 & 3 & 3 & a \\ 10 & -6 & 6 & b \\ 3 & -1 & 3 & c \end{bmatrix}$$

where we have dropped the factor 1/2 (Why?). Row reduction leads to

$$\begin{bmatrix} -1 & 3 & 3 & 0 \\ 0 & 24 & 36 & b+10a \\ 0 & 8 & 12 & c+3a \end{bmatrix}$$

or

$$\begin{bmatrix} -1 & 3 & 3 & a \\ 0 & 2 & 3 & \frac{b+10a}{12} \\ 0 & 8 & 12 & c+3a \end{bmatrix}$$

and finally

$$\begin{bmatrix} -1 & 3 & 3 & a \\ 0 & 2 & 3 & \frac{b+10a}{12} \\ 0 & 0 & 0 & \frac{3c-9a-b}{3} \end{bmatrix}$$

Hence, the vector $\vec{b}$ is a linear combination of the vectors $T(\vec{e}_1), T(\vec{e}_2), T(\vec{e}_3)$ if and only if

$$3c - 9a - b = 0 \ .$$

Hence T is not onto.