**Practice Test 4A for Calculus II, Math 1502, November 9, 2013**

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**PRINT Section:**

**PRINT Name of TA:**

This test is to be taken without calculators and notes of any sorts. The allowed time is 50 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write $1.414\ldots$. Show your work, otherwise credit cannot be given.

**PRINT your name, your section number as well as the name of your TA on EVERY PAGE of this test. This is very important.**

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**I:** For the matrix

$$A = \begin{bmatrix} 1 & 2 & 4 & -19 & 7 \\ 2 & 5 & 5 & -26 & 9 \\ 3 & 6 & 6 & -27 & 9 \end{bmatrix}$$

a) (10 points) Find a basis for the column space of the matrix $A$. Row reduction leads to

$$\begin{bmatrix} 1 & 2 & 4 & -19 & 7 \\ 0 & 1 & -3 & 12 & -5 \\ 0 & 0 & 1 & -5 & 2 \end{bmatrix}$$

The first, second and third column are pivotal columns and hence

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

form a basis for the column space.

b) (10 points) Find a basis for the null space of the matrix $A$.

Calling the variables $v, w, x, y, z$ we find that the variables $y$ and $z$ are free. Solving for the others yields

$$\begin{bmatrix} v \\ w \\ x \\ y \\ z \end{bmatrix} = y \begin{bmatrix} -7 \\ 3 \\ 5 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 3 \\ -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

Hence,

$$\begin{bmatrix} -7 \\ 3 \\ 5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

are a basis for the null space of $A$. Note that the dimensions add up to the number of columns.

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**II:** Consider the subspace $S$ of $\mathcal{R}^3$ spanned by the vectors

$$\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 12 \end{bmatrix}$$

a) (10 points) Find a basis for this subspace.
Again, row reduction of the matrix

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & -2 \\ 2 & 1 & 12 \end{bmatrix}$$

leads to

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

and hence the first two columns are pivotal and

$$\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \vec{v} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

form a basis.
b) (10 points) Now, consider the subspace $T$ of $\mathcal{R}^3$ spanned by the vectors

$$\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix}$$

How are $T$ and $S$ related? Once more row reducing the matrix

$$\begin{bmatrix} 2 & 0 & 5 \\ 1 & -1 & 2 \\ -1 & 5 & 0 \end{bmatrix}$$

leads to

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and the vectors

$$\vec{x} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \vec{y} = \begin{bmatrix} 0 \\ -1 \\ 5 \end{bmatrix}$$

form a basis of $T$. Note that

$$\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \vec{v} - \vec{u}$$

and

$$\begin{bmatrix} 0 \\ -1 \\ 5 \end{bmatrix} = 3\vec{u} - \vec{v} \ .$$

Hence $T = S$. The set of vectors $\{\vec{u}, \vec{v}\}$ and $\{\vec{x}, \vec{y}\}$ are different bases but span the same space.

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**III:** a) (10 points) Compute the inverse of the matrix

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 2 & 2 \\ 3 & 1 & 1 \end{bmatrix} .$$

We have to row reduce

$$\left[\begin{array}{ccc|ccc} 1 & 3 & -1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

which yields the inverse

$$\frac{1}{4}\begin{bmatrix} 0 & -1 & 2 \\ 1 & 1 & -1 \\ -1 & 2 & -1 \end{bmatrix}$$

b) (10 points) Find the third column of the matrix $B^{-1}$ where

$$B = \begin{bmatrix} -1 & -7 & -3 \\ 2 & 15 & 6 \\ 1 & 3 & 2 \end{bmatrix}$$

without computing the other columns.

We have to compute $B^{-1}\vec{e}_3$ which means that we have solve $B\vec{x} = \vec{e}_3$. Hence we have to row reduce

$$\begin{bmatrix} -1 & -7 & -3 & 0 \\ 2 & 15 & 6 & 0 \\ 1 & 3 & 2 & 1 \end{bmatrix}$$

which yields

$$\begin{bmatrix} 1 & 7 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

and back substitution yields

$$\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} .$$

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**IV:** Compute all the eigenvalues and the corresponding eigenvectors of the matrix

$$a)\ (10\text{ points})\quad \begin{bmatrix} 6 & -2 \\ 6 & -1 \end{bmatrix}$$

The characteristic polynomial is

$$\lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3) = 0$$

and hence the eigenvalues are $2, 3$. The eigenvectors for the eigenvalue 2 are any non-zero multiple of the vector

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The eigenvector for the eigenvalue 3 is any non-zero multiple of the vector

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} .$$

b) (10 points) The matrix

$$\begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}$$

has the 1 as an eigenvalue. Find the other eigenvalues and the corresponding eigenvectors.

We have to compute the determinant of the matrix

$$\begin{bmatrix} 2-\lambda & 2 & -1 \\ 1 & 3-\lambda & -1 \\ -1 & -2 & 2-\lambda \end{bmatrix}$$

which yields the characteristic polynomial

$$-\lambda^3 + 7\lambda^2 - 11\lambda + 5 \ .$$

We know that 1 is a root (check!) and hence we can divide

$$-\lambda^3 + 7\lambda^2 - 11\lambda + 5 : (\lambda - 1) = -\lambda^2 + 6\lambda - 5 \ .$$

Solving

$$0 = \lambda^2 - 6\lambda + 5 = (\lambda - 1)(\lambda - 5)$$

and hence the eigenvalues are 1 and 5 where 1 is a double root. To find the eigenvectors for the eigenvalue 1 we have to row reduce the matrix

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 2 & -1 \\ 1 & 2 & -1 \end{bmatrix}$$

which yields

$$\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} , \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

as a basis for the corresponding eigenspace. For the eigenvalue 5 we row reduce the matrix

$$\begin{bmatrix} -3 & 2 & -1 \\ 1 & -2 & -1 \\ -1 & -2 & -3 \end{bmatrix}$$

which yields any nonzero multiple of the vector

$$\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

as eigenvectors.

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**V: (5 points each) Prove or find a counter example.** Let $A$ be an $m \times n$ matrix.

a) If the columns are linearly independent then the matrix is invertible.

This is false. Take the $2 \times 1$ matrix

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

which is clearly not invertible.

b) If the columns are linearly independent and span $\mathcal{R}^m$ then $n = m$.

This is true. Since there are $n$ linearly independent columns we must have $n \leq m$. Since the columns are vectors in $\mathcal{R}^m$ and span $\mathcal{R}^m$ they must be a basis and hence $n = m$.

c) If the dimension of $Nul(A)$ is $n - 1$ then $m = 1$

This is false. take the $2 \times 2$ matrix

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

The null space consists of any multiple of the vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and hence the dimension of the null space is 1 but $m = 2$. The number of columns is NOT necessarily the dimension of the column space. The dimension of the column space is 1 in this example.

d) If the dimension of $Col(A) = n$ then $A$ is invertible.

This is false. Take the $2 \times 1$ matrix

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The dimension of the column space is 1 but the matrix is not invertible.