Homework I, due Thursday January 30

I: (15 points) Exercise on lower semi-continuity: Let X be a normed space and $f : X \to \mathbb{R}$ be a function. We say that f is lower semi - continuous at x_0 if for every $\varepsilon > 0$ there exists $\delta > 0$ so that

$$f(x) - f(x_0) > -\varepsilon$$

whenever $\|x_0 - x\| < \delta$. We also say that f is lower semi-continuous if f is lower semi-continuous at every point of X.

a) Prove that f is lower semi-continuous at x_0 if and only if for every sequence x_n with $\lim_{n\to\infty} x_n = x_0$, it follows that $\liminf_{n\to\infty} f(x_n) \geq f(x_0)$.

b) Prove that f is lower semi-continuous if and only if the set

$$\{x \in X : f(x) > t\}$$

is open for every value of t.

II: (20 points) The space $H^1(\Omega)$ let $\Omega \subset \mathbb{R}^n$ be an open set and consider $L^2(\Omega, dx)$. Denote by $C_c^\infty(\Omega)$ the set of infinitely differentiable functions on Ω that have compact support. Note that, by definition, a continuous function has compact support if the closure of the set where f does not vanish is compact. Here are some facts about $L^2(\Omega, dx)$ and $C_c^\infty(\Omega)$: As you know $L^2(\Omega, dx)$ is a Hilbert space with inner product

$$(f, h) = \int_\Omega f(x)\overline{h(x)}dx \ .$$

Another useful fact is that $C_c^\infty(\Omega)$ is dense in $L^2(\Omega, dx)$.

We define the space $H^1(\Omega)$ to consist of all functions $f \in L^2(\Omega, dx)$ with the property that there exist functions $g_f^i \in L^2(\Omega, dx), i = 1 \dots n$ such that

$$\int_\Omega f(x)\frac{\partial \phi}{\partial x_i}(x)dx = -\int_\Omega g_f^i(x)\phi(x)dx \ , i = 1. \dots, n$$

for every $\phi \in C_c^\infty(\Omega)$. The expression

$$(f, h)_1 = \int_{\mathbb{R}} f(x)\overline{h(x)}dx + \sum_{i=1}^n \int_{\mathbb{R}} g_f^i(x)\overline{g_h^i(x)}dx$$

defines obviously an inner product on $H^1(\Omega)$.

a) Prove that the "gradient" $g_f^i, i = 1, \dots, n$ is unique.

b) Prove that $H^1(\Omega)$ is a Hilbert space. (You do not have to check that $(\cdot, \cdot)_1$ is an inner product. Just prove completeness.)

III: (10 points) On $L^2(\mathbb{R})$ consider the sequences $f_j(x) = f(x-j)$ and $g_j(x) = j^{1/2}g(jx)$ where f, g are fixed functions in $L^2(\mathbb{R})$. Show that both sequences converge weakly to zero as $j \to \infty$.

IV: (15 points) Let f_j, g_j be any two strongly convergent sequences in an arbitrary infinite dimensional Hilbert space $\mathcal{H}$ and h_j, k_j any two weakly convergent sequences in $\mathcal{H}$. Prove or find a counterexample:

a) The sequence (f_j, g_j) is always convergent.

b) The sequence (f_j, h_j) is always convergent.

c) The sequence (h_j, k_j) is always convergent.

Here $(\cdot, \cdot)$ denotes the inner product in $\mathcal{H}$.

V: Extra credit (20 points): This exercise is difficult. Let X be a complete normed space (which we assume for simplicity to be real) and assume that the norm satisfies the parallelogram identity, i.e.,

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

for all $x, y \in X$. Prove that X is a Hilbert space, i.e., there exists an inner product (x, y) such that $\|x\| = \sqrt{(x, x)}$.