Homework I, Solutions

I: (15 points) Exercise on lower semi-continuity: Let X be a normed space and $f : X \to \mathbb{R}$ be a function. We say that f is lower semi - continuous at x_0 if for every $\varepsilon > 0$ there exists $\delta > 0$ so that

$$f(x) - f(x_0) > -\varepsilon \tag{1}$$

whenever $\|x_0 - x\| < \delta$. We also say that f is lower semi-continuous if f is lower semi-continuous at every point of X.

a) Prove that f is lower semi-continuous at x_0 if and only if for every sequence x_n with $\lim_{n\to\infty} x_n = x_0$, it follows that $\liminf_{n\to\infty} f(x_n) \geq f(x_0)$.

Assume that f is lower semi-continuous. If $\|x_n - x_0\| \to 0$ as $n \to \infty$ then for any given ε there exists N so that $\|x_n - x_0\| < \delta$ for all $n > N$. Hence if f is lower semi-continuous,

$$f(x_n) > f(x_0) - \varepsilon$$

for all $n > N$ and hence

$$\liminf_{n\to\infty} f(x_n) \geq f(x_0) - \varepsilon \ .$$

Since ε is arbitrary the result follows.

For the converse, assume that there is an x_0 at which f is not lower semi-continuous. This means that there exists $\varepsilon > 0$ so that for any $\delta > 0$, there exists x with $\|x - x_0\| < \delta$ such that

$$f(x) - f(x_0) \leq -\varepsilon \ .$$

This means that there is a sequence x_n converging to x_0 so that

$$f(x_n) - f(x_0) \leq -\varepsilon$$

for all $n = 1, 2, \ldots$. In particular

$$f(x_0 \leq \liminf_{n\to\infty} f(x_n) \leq f(x_0 - \varepsilon \ .$$

A contradiction.

b) Prove that f is lower semi-continuous if and only if the set

$$\{x \in X : f(x) > t\}$$

is open for every value of t.

Assume that f is lower semi-continuous. Pick any $x_0 \in \{x \in X : f(x) > t\}$. Since $f(x_0) > t$, we also have that $f(x_0) > t + \varepsilon$ for ε sufficiently small. By assumption there exists $\delta > 0$ so that $f(x) > f(x_0) - \varepsilon > t$ for all x with $\|x - x_0\| < \delta$. Hence $\{x \in X : f(x) > t\}$ is open.

To see the converse, assume that $\{x \in X : f(x) > t\}$ is open for all $t \in \mathbb{R}$. Suppose that there exists x_0 at which f is not lower semi-continuous. Then there exists $\varepsilon > 0$ so that for all $\delta > 0$ there exists x with $\|x - x_0\| < \delta$ such that

$$f(x) - f(x_0) \leq -\varepsilon \ .$$

Pick $t = f(x_0) - \varepsilon/2$. Then we there exists a sequence x_n converging to x_0 so that

$$f(x_n) \leq f(x_0) - \varepsilon = t - \varepsilon/2$$

which contradicts the fact that $\{x \in X : f(x) > t\}$ is open.

II: (20 points) The space $H^1(\Omega)$: Let $\Omega \subset \mathbb{R}^n$ be an open set and consider $L^2(\Omega, dx)$. Denote by $C_c^\infty(\Omega)$ the set of infinitely differentiable functions on Ω that have compact support. Note that, by definition, a continuous function has compact support if the closure of the set where f does not vanish is compact. Here are some facts about $L^2(\Omega, dx)$ and $C_c^\infty(\Omega)$: As you know $L^2(\Omega, dx)$ is a Hilbert space with inner product

$$(f, h) = \int_\Omega f(x)\overline{h(x)}dx \ .$$

Another useful fact is that $C_c^\infty(\Omega)$ is dense in $L^2(\Omega, dx)$.

We define the space $H^1(\Omega)$ to consist of all functions $f \in L^2(\Omega, dx)$ with the property that there exist functions $g_f^i \in L^2(\Omega, dx), i = 1 \dots n$ such that

$$\int_\Omega f(x)\frac{\partial \phi}{\partial x_i}(x)dx = -\int_\Omega g_f^i(x)\phi(x)dx \ , i = 1. \dots, n \ .$$

The expression

$$(f, h)_1 = \int_{\mathbb{R}} f(x)\overline{h(x)}dx + \sum_{i=1}^n \int_{\mathbb{R}} g_f^i(x)\overline{g_h^i(x)}dx$$

defines obviously an inner product on $H^1(\Omega)$.

a) Prove that the "gradient" $g_f^i, i = 1, \dots, n$ is unique.

Suppose that there exists another 'gradient' $h_f^i, i = 1, \dots, n$. Then we have that

$$\int_\Omega (g_f^i - h_f^i)\phi dx = 0 \ , \ i = 1, \dots, n$$

for all $\phi \in C_c^\infty(\Omega)$. Fix the index i and set $u_i := g_f^i - h_f^i$. The fact that $C_c^\infty(\Omega)$ is dense in $L^2(\Omega, dx)$ means that for any $\varepsilon > 0$ there exists ϕ^i so that

$$\|u_i - \phi^i\| < \varepsilon \ .$$

We shall use the sign $\| \cdot \|$ to denote the norm in $L^2(\Omega)$. Hence

$$0 = \int_\Omega u^i \phi^i dx = \int_\Omega u^i(\phi^i - u^i)dx + \|u^i\|^2$$

so that

$$\|u^i\|^2 = -\int_\Omega u^i(\phi^i - u^i)dx \le \|u^i\|\|u^i - \phi^i\|$$

by Schwarz's inequality. Thus, if $u^i \neq 0$ we have that

$$\|u^i\| \le \|u^i - \phi^i\| < \varepsilon$$

which proves the claim since ε is arbitrary.

b) Prove that $H^1(\Omega)$ is a Hilbert space.

Let f_n be a Cauchy sequence in $H^1(\Omega)$. This means that

$$\|f_n - f_m\|^2 + \sum_{i=1}^n \|g^i_{f_n} - g^i_{f_m}\|^2 \to 0$$

as $n, m \to \infty$. This means that f_n converges to f in $L^2(\Omega)$ and $g^i_{f_n}$ converges to some g^i in $L^2(\Omega)$. It remains to show that

$$\int_\Omega f \frac{\partial \phi}{\partial x_i} dx = -\int_\Omega g^i \phi dx \ .$$

To see this note that

$$\left| \int_\Omega f_n \frac{\partial \phi}{\partial x_i} dx - \int_\Omega f \frac{\partial \phi}{\partial x_i} dx \right| = \left| \int_\Omega [f_n - f] \frac{\partial \phi}{\partial x_i} dx \right| \le \|f - f_n\| \|\frac{\partial \phi}{\partial x_i}\| \to 0$$

as $n \to \infty$. Hence

$$\int_\Omega f \frac{\partial \phi}{\partial x_i} dx = \lim_{n\to\infty} \int_\Omega f_n \frac{\partial \phi}{\partial x_i} dx = -\lim_{n\to\infty} \int_\Omega g^i_{f_n} \phi dx = -\int_\Omega g^i \phi dx \ ,$$

where we used that

$$\left| \int_\Omega g^i_{f_n} \phi dx - \int_\Omega g^i \phi dx \right| \le \|g^i_{f_n} - g^i\| \|\phi\| \ .$$

III: (10 points) On $L^2(\mathbb{R})$ consider the sequences $f_j(x) = f(x - j)$ and $g_j(x) = j^{1/2} g(jx)$ where f, g are fixed functions in $L^2(\mathbb{R})$. Show that both sequences converge weakly to zero as $j \to \infty$.

The idea is that for a function $h \in L^2(\mathbb{R})$ the sequence (f_j, h) tends to zero since the overlap between f_j and h tends to disappear. The problem is that the functions under consideration are only in $L^2(\mathbb{R})$ and they do not have compact support. To really prove this one argues via approximations. Pick any $\varepsilon > 0$. There exists $\phi, \psi \in C^\infty_c(\mathbb{R})$ so that $\|f - \psi\| < \varepsilon$ and $\|h - \phi\| < \varepsilon$. If we set $\psi_j(x) = \psi(x - j)$ we have that $\|f_j - \psi_j\| = \|f - \psi\|$ by changing variables in the integration. Hence

$$\begin{aligned} |(f_j, h)| = |(f_j - \psi_j, h) + (\psi_j, h - \phi) + (\psi_j, \phi)| &\le \|f_j - \psi_j\| \|h\| + \|\psi_j\| \|h - \phi\| + |(\psi_j, \phi)| \\ &\le \varepsilon(\|h\| + \|\psi\|) + |(\psi_j, \phi)| \end{aligned}$$

For j large enough $|(\psi_j, \phi)| = 0$ since the two functions ϕ and ψ have compact support and then the supports of ϕ and $\psi(x - j)$ do not overlap. Thus for j sufficiently large

$$|(f_j, h)| \le \varepsilon(\|h\| + \|\psi\|)$$

and since ε is arbitrary this means that $\lim_{j\to\infty} |(f_j, h)| = 0$.

For the second problem we proceed exactly the same way except that we set $\psi_j(x) = j^{1/2}\psi(jx)$ so that $\|g_j - \psi_j\| = \|g - \psi\|$. Hence

$$|(g_j, h)| \le \varepsilon(\|h\| + \|\psi\|) + |(\psi_j, \phi)|$$

Now

$$|(\psi_j, \phi)| \le j^{1/2} \int |\psi(jx)||\phi(x)| dx \le C j^{1/2} \int |\psi(jx)| dx = C j^{-1/2} \int |\psi(x)| dx$$

where $C = \max_x |\phi(x)|$. The rest follows as before.

IV (15 points) : Let f_j, g_j be any two strongly convergent sequences in an arbitrary infinite dimensional Hilbert space $\mathcal{H}$ and h_j, k_j any two weakly convergent sequences in $\mathcal{H}$. Prove or find a counterexample:

a) The sequence (f_j, g_j) is always convergent.
Yes. Assume that f_j resp. g_j converge strongly to f resp. g. Then

$$|(f_j, g_j) - (f, g)| = |(f_j - f, g_j) + (f, g_j - g)| \le \|f - f_j\|\|g\| + \|g - g_j\|\|f_j\|$$

which converges to zero since $\|f_j\|$ stays bounded.

b) The sequence (f_j, h_j) is always convergent.
Yes. Suppose that h_j converges weakly to h. Then

$$|(f_j, h_j) - (f, h)| = |(f_j - f, h_j) + (f, h_j - h)| \le \|f - f_j\|\|h_j\| + |(f, h_j - h)| \to 0$$

as $n \to \infty$ since $\|h_j\|$ is bounded by the uniform boundedness principle.

c) The sequence (h_j, k_j) is always convergent.
No. Take h_j to be an orthonormal sequence, $(h_j, h_k) = \delta_{j,k}$ and define $k_j = h_j$ for j even and $k_j = 0$ for j odd. Both sequences converge weakly to zero. But (h_j, k_j) is a sequence that alternates between 0 and 1 and hence does not converge.

Here $(\cdot, \cdot)$ denotes the inner product in $\mathcal{H}$.

V: Extra credit: This exercise is difficult. Let X be a complete normed space (which we assume for simplicity to be real) and assume that the norm satisfies the parallelogram identity, i.e.,

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

for all $x, y \in X$. Prove that X is a Hilbert space, i.e., there exists an inner product (x, y) such that $\|x\| = \sqrt{(x, x)}$.

Solution: We *define*

$$f(x, y) = \frac{1}{2}(\|x + y\|^2 - \|x\|^2 - \|y\|^2)$$

First we show that $f(x, y)$ is *additive* in each variable. Clearly f is symmetric and hence it suffices to do this for the first variable. We have to show that

$$0 = f(x + y, z) - f(x, z) - f(y, z)$$

$$= \frac{1}{2}(\|x + y + z\|^2 - \|x + y\|^2 - \|x + z\|^2 - \|y + z\|^2 + \|x\|^2 + \|y\|^2 + \|z\|^2$$

Now set

$$a = \frac{x + y}{2}\ ,\ b = \frac{x + z}{2}\ ,\ c = \frac{y + z}{2}\ .$$

Using this notation we have to show that

$$\|a + b + c\|^2 + \|a + b - c\|^2 + \|a + c - b\|^2 + \|b + c - a\|^2 = 4\|a\|^2 + 4\|b\|^2 + 4\|c\|^2\ .$$

Now using the parallelogram identity

$$\|a+b+c\|^2 + \|a+b-c\|^2 = 2\|a+b\|^2 + 2\|c\|^2$$

and

$$\|a+c-b\|^2 + \|b+c-a\|^2 = \|c+(a-b)\|^2 + \|c-(a-b)\|^2 = 2\|c\|^2 + 2\|a-b\|^2 \ ,$$

and

$$2\|a+b\|^2 + 2\|c\|^2 + 2\|c\|^2 + 2\|a-b\|^2 = 4\|a\|^2 + 4\|b\|^2 + 4\|c\|^2 \ .$$

It remains to show that $f(x,y)$ is *homogeneous* in each variable. By symmetry it suffices to show this for the first variable. If p is any integer we have by induction that

$$f(px,y) = pf(x,y)$$

and for $q \neq 0$ an integer

$$qf(\frac{x}{q},y) = f(x,y) \text{ or } f(\frac{x}{q},y) = \frac{1}{q}f(x,y)$$

so that

$$f(rx,y) = rf(x,y)$$

for any rational number r. The function $x \to f(x,y)$ is continuous. To see this note that

$$|f(x_1,y) - f(x_2,y)| = \frac{1}{2}\left|\|x_1+y\|^2 - \|x_1\|^2 - \|x_2+y\|^2 + \|x_2\|^2\right|$$

$$= \frac{1}{2}(\|x_1+y\| + \|x_2+y\|)(\|x_1+y\| - \|x_2+y\|) - \frac{1}{2}(\|x_1\| + \|x_2\|)(\|x_2\| - \|x_1\|)$$

$$\leq (\|x_1\| + \|x_2\| + \|y\|)\,\|x_2 - x_1\|$$

by the triangle inequality.

For c real, pick a sequence r_n of rational numbers that converge to c. Clearly

$$cf(x,y) = \lim_{n\to\infty} r_n f(x,y) = \lim_{n\to\infty} f(r_n x,y) = f(cx,y) \ .$$

Hence, $f(x,y)$ satisfies the conditions of an inner product.