Homework II, due Tuesday February 25

I: easy (10 points) Let $\mathcal{H}_1, \mathcal{H}_2$ be two Hilbert spaces and $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator.

a) Prove that there exists a unique linear operator $A^* : \mathcal{H}_2 \to \mathcal{H}_1$ with

$$\langle Af, g\rangle_2 = \langle f, A^* g\rangle_1$$

where $\langle \cdot, \cdot\rangle_1, \langle \cdot, \cdot\rangle_2$ denote the inner products in $\mathcal{H}_1$ resp. $\mathcal{H}_2$.

b) Prove that $\|A\| = \|A^*\|$.

II: (20 points) Let T be a bounded operator on a separable Hilbert space $\mathcal{H}$. Assume that for some orthonormal basis $f_1, f_2, f_3, \ldots$

$$\sup_{\|g\|=1, g\perp[f_1,f_2,\ldots,f_n]} \|Tg\| \to 0$$

as $n \to \infty$. Here $[f_1, \ldots, f_n]$ denotes the span of the vectors $f_1, \ldots, f_n$. Prove that T is compact.

III: (35 points) Suppose that $T : \mathcal{H} \to \mathcal{H}$ is a compact operator. Let $\lambda \neq 0$ be a complex number.

a) Show that for any $g \in \mathrm{Ran}(T - \lambda I)$ among all the solutions of the equation

$$(T - \lambda I)f = g$$

there exists a solution f_0 that has least length.

b) Show that there exists a constant C independent of $g \in \mathrm{Ran}(T - \lambda I)$ so that the least length solution satisfies

$$\|f_0\| \leq C\|g\|$$

c) Use b) to show that $\mathrm{Ran}(T - \lambda I)$ is closed.

IV: (35 points) Let $A : \mathcal{H} \to \mathcal{H}$ be a linear operator and A^* its adjoint. Assume further that

$$AA^* - A^*A = I \ .$$

Prove that A^* and hence A cannot be bounded operators on $\mathcal{H}$. Proceed in the following way:

a) Guess first a formula for $A^n A^{*n} f$ in terms of lower powers of A and A^* and then prove it. (Hint: Compute

$$\left(\frac{d}{dx}\right)^n (x^n h(x))$$

where $h(x)$ is a smooth function on the real line.)

b) Use a) to show that $\|A^{*n} f\| \geq \sqrt{n!}\|f\|$ and deduce from this that A^* cannot be bounded.

V: (30 points) Extra credit It is not difficult to prove that the space $C_c^\infty(\mathbb{R})$ is dense in $H^1(\mathbb{R})$, so we assume this fact as given. Not that for $f \in H^1(\mathbb{R}), x_0 \in \mathbb{R}$ it does *not* make sense to talk about $f(x_0)$ since such functions are only defined almost everywhere. $f(x_0)$, however, is defined for $f \in C_c^\infty(\mathbb{R})$ and the functional $f \to f(x_0)$ is abviously linear which we denote by $\ell_{x_0}(f)$.

a) Prove that for any function $f \in C_c^\infty(\mathbb{R})$

$$\max |f(x)|^2 \leq \left(\int_{\mathbb{R}} |f'(x)|^2 dx \right)^{1/2} \left(\int_{\mathbb{R}} |f(x)|^2 dx \right)^{1/2}$$

b) Prove that $\ell_{x_0}(f)$ can be uniquely extended to a bounded linear functional on $H^1(\mathbb{R})$, which is called the **trace** of the function $f \in H^1(\mathbb{R})$.

c) Find $g_{x_0} \in H^1(\mathbb{R})$ so that

$$\ell_{x_0}(f) = \langle f, g_{x_0} \rangle_{H^1(\mathbb{R})} \ .$$

(Hint: Think of the Green function.)