Homework II, due Tuesday February 25

I: easy (10 points) Let $\mathcal{H}_1, \mathcal{H}_2$ be two Hilbert spaces and $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator.

a) Prove that there exists a unique linear operator $A^* : \mathcal{H}_2 \to \mathcal{H}_1$ with

$$\langle Af, g\rangle_2 = \langle f, A^* g\rangle_1$$

where $\langle \cdot, \cdot\rangle_1, \langle \cdot, \cdot\rangle_2$ denote the inner products in $\mathcal{H}_1$ resp. $\mathcal{H}_2$.

Simply note that the functional

$$f \to \langle Af, g\rangle_2$$

is a linear function on $\mathcal{H}_1$ and it is bounded sinnce A is bounded, by Schwarz's inequality. By the Riesz representation theorem there exists a unique element $h \in \mathcal{H}_1$ such that

$$\langle Af, g\rangle_2 = \langle f, h\rangle_1$$

for all $f \in \mathcal{H}_1$. It is easy to see that the map $g \to h$ is linear and hence we may define $h = A^* g$.

b) Prove that $\|A\| = \|A^*\|$.

This follows from

$$\|A\| = \sup_{\|f\|_1 = \|g\|_2 = 1} |\langle Af, g\rangle_2| = \sup_{\|f\|_1 = \|g\|_2 = 1} |\langle f, A^* g\rangle_1| = \|A^*\| \ .$$

II: (20 points) Let T be a bounded operator on a separable Hilbert space $\mathcal{H}$. Assume that for some orthonormal basis $f_1, f_2, f_3, \dots$

$$\sup_{\|g\|=1, g\perp[f_1, f_2, \dots, f_n]} \|Tg\| \to 0$$

as $n \to \infty$. Here $[f_1, \dots, f_n]$ denotes the span of the vectors $f_1, \dots, f_n$. Prove that T is compact.

Consider the operator

$$T_N f = \sum_{k=1}^{N} T f_i \langle f, f_i\rangle$$

so that

$$(T - T_N) f = \sum_{k=N+1}^{\infty} T f_i \langle f, f_i\rangle$$

Thus, for $f \in [f_1, \dots, f_N]$

$$(T - T_N) f = 0 \ .$$

Let $h \in \mathcal{H}$ arbitrary. We can write $h = h_1 + h_2$ where $h_1 \in [f_1, \dots, f_N]$ and $h_2 \in [f_1, \dots, f_N]^\perp$. Thus

$$(T - T_N) h = T h_2$$

and hence

$$\|(T - T_N)h\| = \|Th_2\| \le \sup_{\|g\|=1, g\perp[f_1, f_2, \dots, f_n]} \|Tg\| \|h_2\| \le \|Th_2\| \le \sup_{\|g\|=1, g\perp[f_1, f_2, \dots, f_n]} \|Tg\| \|h\|$$

Therefore, by our assumption, $\|T - T_N\| \to 0$ as $N \to \infty$. Since T_N is compact for any finite N, T is also compact.

III: (35 points) Suppose that $T : \mathcal{H} \to \mathcal{H}$ is a compact operator. Let $\lambda \neq 0$ be a complex number.

a) Show that for any $g \in \mathrm{Ran}(T - \lambda I)$ among all the solutions of the equation

$$(T - \lambda I)f = g$$

there exists a solution f_0 that has least length.

Set

$$D = \inf\{\|f\| : (T - \lambda I)f = g\}$$

and consider the a minimizing sequence f_n with $(T - \lambda I)f_n = g$ and $\|f_n\| \to D$. Since the sequence is bounded there exists a weakly convergent subsequence, which we denote again by f_n. Let f be the weak limit. Since

$$Tf_n - \lambda f_n = g$$

for all n and since T is compact we know that Tf_n converges strongly and hence f_n converges strongly to f. This implies that $\|f\| = D$ and since T is continuous we find that $(T - \lambda)f = g$.

b) Show that there exists a constant C independent of $g \in \mathrm{Ran}(T - \lambda I)$ so that the least length solution satisfies

$$\|f_0\| \leq C\|g\|$$

Suppose there is not such a constant. Then we could find a sequence g_n so that $\|g_n\| \to 0$ while the least length solution of $(T - \lambda)f_n = g_n$ satisfies $\|f_n\| = 1$. Once more, we can pass to a weakly convergent subsequence and can therefore assum that f_n converges weakly to some function f. Since T is compact, Tf_n converges strongly to Tf and sinceg_n converges strongly to zero we find that f_n converges strongly to f and hence

$$Tf - \lambda f = 0 \ .$$

This implies that

$$(T - \lambda I)(f_n - f) = g_n$$

and for n large enough $\|f_n - f\| < 1$ which contradicts the fact that f_n is the least length solution.

c) Use b) to show that $\mathrm{Ran}(T - \lambda I)$ is closed. This is immediate. Let g_n be in the range of $T - \lambda I$, i.e, $(T - \lambda I)f_n = g_n$ and where we can assume that f_n is the least length solution and $g_n \to g$. Since there exists C fixed so that

$$\|f_n - f_m\| \leq C\|g_n - g_m\|$$

we see that f_n is a Cauchy sequence. Hence $f_n \to f$ and $(T - \lambda I)f = g$ so that g is in the rnage of $T - \lambda I$.

IV: (35 points) Let $A : \mathcal{H} \to \mathcal{H}$ be a linear operator and A^* its adjoint. Assume further that

$$AA^* - A^*A = I \ .$$

Prove that A^* and hence A cannot be bounded operators on $\mathcal{H}$. Proceed in the following way:

a) Guess first a formula for $A^n {A^*}^n f$ in terms of lower powers of A and A^* and then prove it. (Hint: Compute

$$\left(\frac{d}{dx}\right)^n (x^n h(x))$$

where $h(x)$ is a smooth function on the real line.)

The Leibniz rule yields

$$\sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) (x^n)^{(k)} h^{(n-k)} = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} \frac{n!}{(n-k)!} x^{n-k} h^{(n-k)} \ .$$

Since $\frac{d}{dx}x - x\frac{d}{dx} = 1$, which are the same commutation relations that A and A^* satisfies, we find

$$A^n A^{*n} = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} \frac{n!}{(n-k)!} {A^*}^{n-k} A^{n-k} \ .$$

One easily checks that the formula holds for $n = 1$, since

$$AA^* = A^*A + I \ .$$

Assume the formula for n and compute

$$A^{n+1} A^{*n+1} = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} \frac{n!}{(n-k)!} A{A^*}^{n-k} A^{n-k} A^* \ .$$

Now

$$AA^{*n-k} = A^{*n-k}A + (n-k)A^{*n-k-1} \ ,$$

and hence we get

$$AA^{*n-k}A^{n-k}A^* = A^{*n-k}A^{n-k+1}A^* + (n-k)A^{*n-k-1}A^{n-k}A^* \ .$$

Further,

$$A^{n-k+1}A^* = A^*A^{n-k+1} + (n-k+1)A^{n-k}$$

and

$$A^{n-k}A^* = A^*A^{n-k} + (n-k)A^{n-k-1} \ ,$$

so that

$$AA^{*n-k}A^{n-k}A^* = A^{*n-k+1}A^{n-k+1} + (2(n-k)+1)A^{*n-k}A^{n-k} + (n-k)^2A^{*n-k-1}A^{n-k-1} \ .$$

Using this and changing indices we get

$$A^{n+1}A^{*n+1} = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} \frac{n!}{(n-k)!} A^{*n+1-k}A^{n+1-k}$$

$$+ \sum_{k=1}^{n+1} \frac{n!}{(n-k+1)!(k-1)!} \frac{n!}{(n-k+1)!} [2(n+1-k)+1] A^{*n+1-k}A^{n+1-k}$$

$$\sum_{k=2}^{n+1} \frac{n!}{(n-k+2)!(k-2)!} \frac{n!}{(n-k+2)!} (n-k+2)^2 A^{*n+1-k}A^{n+1-k}$$

For $2 \leq k \leq n$ we find

$$\frac{n!}{(n-k)!k!} \frac{n!}{(n-k)!} + \frac{n!}{(n-k+1)!(k-1)!} \frac{n!}{(n-k+1)!} [2(n+1-k)+1]$$

$$+ \frac{n!}{(n-k+2)!(k-2)!} \frac{n!}{(n-k+2)!} (n-k+2)^2$$

$$= \frac{(n+1)!}{(n+1-k)!k!} \frac{(n+1)!}{(n+1-k)!}$$

For $k = 1$ the contributions to the coefficients are

$$(2n+1) + n^2 = (n+1)^2 = \frac{(n+1)!}{(n+1-1)!1!}\frac{(n+1)!}{(n+1-1)!}$$

and the term $k = 0$ has the coefficient 1. Hence we obtain the sum

$$A^{n+1}A^{*n+1} = \sum_{k=0}^{n+1} \left(\begin{array}{c} n+1 \\ k \end{array} \right) \frac{(n+1)!}{(n+1-k)!} A^{*n+1-k}A^{n+1-k}$$

which was to be shown.

Hence

$$\|A^{*n}f\|^2 = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!}\frac{n!}{(n-k)!}\|A^{n-k}f\|^2$$

b) Use a) to show that $\|A^{*n}f\| \geq \sqrt{n!}\|f\|$ and deduce from this that A^* cannot be bounded.

This inequality follows by taking the term $k = 0$ which shows that A^* cannot be bounded.

V: (30 points) Extra credit It is not difficult to prove that the space $C_c^\infty(\mathbb{R})$ is dense in $H^1(\mathbb{R})$, so we assume this fact as given. Not that for $f \in H^1(\mathbb{R}), x_0 \in \mathbb{R}$ it does *not* make sense to talk about $f(x_0)$ since such functions are only defined almost everywhere. $f(x_0)$, however, is defined for $f \in C_c^\infty(\mathbb{R})$ and the functional $f \to f(x_0)$ is abviously linear which we denote by $\ell_{x_0}(f)$.

a) Prove that for any function $f \in C_c^\infty(\mathbb{R})$

$$\max |f(x)|^2 \leq \left(\int_{\mathbb{R}} |f'(x)|^2 dx \right)^{1/2} \left(\int_{\mathbb{R}} |f(x)|^2 dx \right)^{1/2}$$

Start with

$$f(x)^2 = 2\int_{-\infty}^{x} f'(y)f(y)dy \leq 2\int_{-\infty}^{x} |f(y)||f'(y)|dy$$

and

$$f(x)^2 = -2\int_{x}^{\infty} f'(y)f(y)dy \leq 2\int_{x}^{\infty} |f(y)||f'(y)|dy$$

Adding these two inequalities yields

$$2f(x)^2 \leq 2\int_{-\infty}^{\infty} |f(y)||f'(y)|dy$$

Now the inequality follows from Schwarz's inequality.

b) Prove that $\ell_{x_0}(f)$ can be uniquely extended to a bounded linear functional on $H^1(\mathbb{R})$, which is called the **trace** of the function $f \in H^1(\mathbb{R})$.

This is immediate from b): We have fro $f \in C_c^\infty(\mathbb{R})$

$$\ell_{x_0}(f) = |f(x_0)| \leq \sqrt{\|f\|\|f'\|} \leq \frac{1}{\sqrt{2}}\sqrt{\|f\|^2 + \|f'\|^2} = \frac{1}{\sqrt{2}}\|f\|_{H^1(\mathbb{R})} \ .$$

Thus ℓ_{x_0} extends uniquely to a bounded linear functional on $H^1(\mathbb{R})$. Further, by the Riesz representation theorem there exists a unique function g so that

$$\ell_{x_0}(f) = \langle f, g \rangle_{H^1(\mathbb{R})} \ .$$

c) Find $g_{x_0} \in H^1(\mathbb{R})$ so that

$$\ell_{x_0}(f) = \langle f, g_{x_0} \rangle_{H^1(\mathbb{R})} \ .$$

(Hint: Think of the Green function.) For $f \in C_c^\infty(\mathbb{R})$ we have to find $g \in H^1(\mathbb{R})$ so that

$$f(x_0) = \int_{\mathbb{R}} [gf + g'f']dx$$

Pick any function $f \in C_c^\infty(-\infty, x_0)$ and note that

$$0 = \int_{\mathbb{R}} [gf + g'f']dx = \int_{\mathbb{R}} [g - g'']f dx$$

from which we conclude that $g = g''$ on the interval $(-\infty, x_0)$. Similarly we also have that $g'' = g$ on (x_0, ∞). Now for general $f \in C_c^\infty(\mathbb{R})$ we find using integration by parts

$$f(x_0) = \int_{\mathbb{R}} [gf+g'f']dx = \lim_{\varepsilon \to 0} [-\int_{-\infty}^{x_0-\varepsilon} [g-g'']f dx + g'(x_0-\varepsilon)f(x_0-\varepsilon) - \int_{x_0+\varepsilon}^{\infty} [g-g'']f dx - f(x_0+\varepsilon)g'(x_0+\varepsilon)] \ ,$$

$$= \lim_{\varepsilon \to 0} [g'(x_0 - \varepsilon) - g'(x_0 + \varepsilon)]f(x_0) \ .$$

Thus, we have to find a function that satisfies $g'' = g$ on $(-\infty, x_0)$ and (x_0, ∞) and its derivative has a jump at x_0, i.e.,

$$\lim_{\varepsilon \to 0} [g'(x_0 - \varepsilon) - g'(x_0 + \varepsilon) = 1 \ .$$

Clearly, the function which has this property is given by

$$ce^{-|x-x_0|}$$

where c is a constant.

$$\lim_{\varepsilon \to 0} [g'(x_0 - \varepsilon) - g'(x_0 + \varepsilon) = 2c \ .$$

abd hence $c = \frac{1}{2}$. To summarize, we find that

$$\int_{-\infty}^{\infty} [f(y)g_{x_0}(y) + f'(y)g'_{x_0}(y)]dy = f(x_0)$$

for all $f \in C_c^\infty(\mathbb{R})$ where

$$g_{x_0}(y) = \frac{1}{2} e^{-|y-x_0|} \ .$$