Homework III, due Thursday March 27

I: (20 points) Let $\mathcal{H}$ be a Hilbert space. A bounded linear operator $A : \mathcal{H} \to \mathcal{H}$ is called **normal** if $AA^* = A^*A$. If A is normal, show the set

$$\{p(A, A^*) : p(w, z) \text{ a polynomial in two variables}\}$$

is a commutative subalgebra of $\mathcal{L}(\mathcal{H})$ which is closed under $*$ and hence its closure is a commutative C^* subalgebra of $\mathcal{L}(\mathcal{H})$.

II: (20 points) Recall that the spectral radius of a bounded linear operator A is defined by $r(A) = \sup_{z \in \sigma(A)} |z|$. We know from the lecture that $r(A) = \lim_{n\to\infty} \|A^n\|^{1/n}$. Show that if A is normal then

$$r(A) = \|A\| \ .$$

(Hint: Prove first that $\|A^2\| = \|A\|^2$.)

III: (20 points) Let A be a normal operator. Prove that for any polynomial $p(\lambda)$

$$\|p(A)\| = \sup_{\lambda \in \sigma(A)} |p(\lambda)| \ .$$

IV: (20 points) a) Show that a normal operator A can be written as

$$A = B + iC$$

where B, C are bounded self adjoint operators which commute, i.e., $BC = CB$.

b) If A is a normal operator then the spectrum

$$\sigma(A) = \{s + it : s \in \sigma(B),\ t \in \sigma(C)\} \ .$$

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V: (20 points) (Taken from Reed-Simon) Suppose that f is a bounded measurable function, but $f \notin L^2(\mathbb{R})$. On the domain

$$D = \{\phi \in L^2(\mathbb{R}) : \int_{\mathbb{R}} |f(x)\phi(x)| dx < \infty\}$$

consider the operator

$$A\phi(x) = \langle \phi, f\rangle \psi \ ,$$

where ψ is a fixed function in $L^2(\mathbb{R})$ and $\langle \cdot, \cdot \rangle$ is the usual inner product. Show that

a) A is densely defined.

b)that $D(A^*)$ is not dense and that on $D(A^*)$ the operator A^* is the zero operator.