Homework III Solutions

I: (20 points) Let $\mathcal{H}$ be a Hilbert space. A bounded linear operator $A : \mathcal{H} \to \mathcal{H}$ is called **normal** if $AA^* = A^*A$. If A is normal, show the set

$$\{p(A, A^*) : p(w, z) \text{ a polynomial in two variables}\}$$

is a commutative subalgebra of $\mathcal{L}(\mathcal{H})$ which is closed under $*$ and hence its closure is a commutative C^* subalgebra of $\mathcal{L}(\mathcal{H})$.

The set of such operators is obviously an algebra. If

$$p(w, z) = \sum c_{j,k} w^j z^k$$

then

$$p(A, A^*) = \sum c_{j,k} A^j A^{*k}$$

and

$$p(A, A^*)^* = \sum \overline{c_{j,k}} A^k A^{*j}$$

which is again in the algebra. The commutativity follows from the assumption that A is normal.

II: (20 points) Let $\mathcal{H}$ be a Hilbert space. A bounded linear operator $A : \mathcal{H} \to \mathcal{H}$ is called **normal** if $AA^* = A^*A$. Show that if A is normal then

$$r(A) = \|A\|$$

where $r(A)$ is the spectral radius of A. (Hint: Prove first that $\|A^2\| = \|A\|^2$.)

We have that

$$\|A\|^2 = \sup_{\|f\|=1} \langle f, A^*Af \rangle = \|A^*A\|$$

since A^*A is self adjoint. Next,

$$\|A^*A\|^2 = \sup_{\|f\|=1} \langle A^*Af, A^*Af \rangle = \sup_{\|f\|=1} \langle A^2 f, A^2 f \rangle = \|A^2\|^2$$

since A is normal. Thus, $\|A^2\| = \|A\|^2$. Since A^2 is also normal we find $\|A^4\| = \|A\|^4$ and, by induction, $\|A^{2^m}\| = \|A\|^{2^m}$. Hence

$$\|A^{2^m}\|^{\frac{1}{2^m}} = \|A\|$$

and since $\|A^n\|^{frac1n}$ converges to $r(A)$, we have that $r(A) = \|A\|$.

III: (20 points) Let A be a normal operator. Prove that for any polynomial $p(\lambda)$

$$\|p(A)\| = \sup_{\lambda \in \sigma(A)} |p(\lambda)| \ .$$

Recall that since $p(A)$ is normal

$$\sup_{z \in \sigma(p(A))} |z| =: r(p(A)) = \|p(A)\|$$

We know, however, from the lecture that

$$\sigma(p(A)) = p(\sigma(A))$$

and hence

$$\sup_{z\in\sigma(p(A))} |z| = \sup_{z\in\sigma(A)} |p(z)| \ .$$

IV: (20 points) a) Show that a normal operator A can be written as

$$A = B + iC$$

where B, C are bounded self adjoint operators which commute, i.e., $BC = CB$.

Define

$$B = \frac{1}{2}(A + A^*) \ , \ C = \frac{1}{2i}(A - A^*)$$

and note that $B^* = B, C^* = C$. This is general and has nothing to do with A being normal. A normal implies that $BC = CB$. Obviously

$$B + iC = A \ .$$

b) If A is a normal operator then the spectrum

$$\sigma(A) \subset \{s + it : s \in \sigma(B), \ t \in \sigma(C)\} \ .$$

.

This is a bit more tricky. Let $\mu \in \rho(B)$. We have to show that $\mu + is \in \rho(A)$ for any $s \in \mathbb{R}$. The operator

$$Q := (B - \mu I)^2 + (C - sI)^2$$

is self adjoint. Since $B - \mu I$ has a bounded inverse we have that

$$\|(B - \mu I)f\|^2 \geq c\|f\|^2$$

for some positive constant c. Hence, $\sigma(Q) \subset (c, \|Q\|)$ and it follows from the spectral theorem that Q is invertible with a bounded inverse. Next consider

$$(B - \mu I - i(C - sI))Q^{-1} = Q^{-1}(B - \mu I - i(C - sI))$$

and note that

$$(A - (\mu + is)I)(B - \mu I - i(C - sI))Q^{-1} = QQ^{-1} = I$$

and

$$Q^{-1}(B - \mu I - i(C - sI)))(A - (\mu + is)) = Q^{-1}Q = I$$

and hence $\mu + is \in \rho(A)$. Likewies, in a similar fashion one can see that if $\nu \in \rho(C)$ then $t + i\nu \in \rho(A)$ for all $t \in \mathbb{R}$. Hence

$$\{\mu + is : \nu \in \rho(B), s \in \mathbb{R}\} \cup \{t + i\nu : t \in \mathbb{R}, \nu \in \rho(C)\} \subset \rho(A) \ .$$

and by taking complements the result follows.

V: (20 points) (Taken from Reed-Simon) Suppose that f is a bounded measurable function, but $f \notin L^2(\mathbb{R})$. On the domain

$$D = \{\phi \in L^2(\mathbb{R}) : \int_{\mathbb{R}} |f(x)\phi(x)|dx < \infty\}$$

consider the operator

$$A\phi(x) = \langle \phi, f \rangle \psi \ ,$$

where ψ is a fixed function in $L^2(\mathbb{R})$ and $\langle \cdot, \cdot \rangle$ is the usual inner product. Show that

a) A is densely defined.

b)that $D(A^*)$ is not dense and that on $D(A^*)$ the operator A^* is the zero operator.

The domain D is dense, since $\mathcal{S}(\mathbb{R}) \subset D$. Recall that $g \in D(A^*)$ if

$$\phi \to \langle A\phi, g \rangle$$

extends to a bounded linear functional on all of $L^2(\mathbb{R})$. But in the example at hand

$$\langle A\phi, g \rangle = \langle \phi, f \rangle \langle \psi, g \rangle \ .$$

The functional

$$\langle \phi, f \rangle$$

is not bounded on $L^2(\mathbb{R})$ since $f \notin L^2(\mathbb{R})$. More precisely, consider the sequence of functions

$$\phi_n(x) = \frac{f(x)\chi_{(-n,n)}(x)}{\sqrt{\int_{-n}^{n} |f(x)|^2 dx}}$$

where $\chi_{(-n,n)}(x)$ is the characteristic function of the interval $(-n, n)$. Clearly $\|\phi_n| = 1$ but

$$\langle \phi_n, f \rangle = \sqrt{\int_{-n}^{n} |f(x)|^2 dx}$$

which diverges as $n \to \infty$. Thus, the domain of A^* consists of all functions g with $\langle \psi, g \rangle = 0$ and on this domain A^* is the zero operator.