Homework IV, due Thursday April 10

I: (20 points) a) Let A be a linear operator on an n- dimensional space and assume that v is a cyclic vector. Show that $v, Av, \cdots, A^{n-1}v$ are linearly independent.

b) Show that a self adjoint operator on a finite dimensional space has a cyclic vector if and only if its eigenvalues have multiplicity one. (The Vandermonde determinant might be useful here).

II: (20 points) (taken from Reed and Simon) a) Let C be a symmetric operator, $A \subset C$ and assume that $\mathrm{Ran}(A + iI) = \mathrm{Ran}(C + iI)$. Show that $A = C$.

b) Let A be a symmetric operator such that $\mathrm{Ran}(A + iI) = \mathcal{H}$ but $\mathrm{Ran}(A - iI) \neq \mathcal{H}$. Show that A does not have a self adjoint extension.

III: (30 points) Let $U : \mathcal{H} \to \mathcal{H}$ be a unitary operator and assume $U - I$ is injective, i.e., 1 is not an eigenvalue of U.

a) Prove that $\mathrm{Ran}(U - I)$ is dense in $\mathcal{H}$.

b) Consider the mean

$$V_N = \frac{1}{N} \sum_{n=0}^{N-1} U^n \ .$$

Prove that for any $f \in \mathcal{H}$

$$\lim_{N \to \infty} \|V_N f\| = 0 \ .$$

IV: (20 points) Let $U : \mathcal{H} \to \mathcal{H}$ be a unitary operator and form the mean

$$V_N = \frac{1}{N} \sum_{n=0}^{N-1} U^n \ .$$

Show that for any $f \in \mathcal{H}$

$$\lim_{N \to \infty} \|V_N f - Pf\| = 0$$

where P is the projection onto the eigenspace of U with eigenvalue 1.