1. Basic theorem on self adjointness

The following theorem is basic to the theory of self adjoint operators. It clarifies the role played by the adjoint of a symmetric operator.

Theorem 1.1. *Let A be a symmetric operator on a Hilbert space $\mathcal{H}$, i.e., A is densely defined and for all $f, g \in D(A)$*

$$\langle Af, g\rangle = \langle f, Ag\rangle \ .$$

Then the following three statements are equivalent, i.e., each of them implies the other two.

a) A is self adjoint,

b) A is closed and $\mathrm{Ker}(A^* \pm iI) = \{0\}$,

c) $\mathrm{Ran}(A \pm iI) = \mathcal{H}$.

Proof. We assume that $A = A^*$ and prove b). Since A^* is closed so is A. Since a self adjoint operator has only realy eigenvalues, $\mathrm{Ker}(A^* \pm iI) = \{0\}$. Next we assume b) and prove c). The range of $(A + iI)$ is dense, for if $f \perp \mathrm{Ran}(A + iI)$ then

$$\langle (A + iI)g, f\rangle = 0$$

for all $g \in D(A)$ and hence

$$\langle Ag, f\rangle = -i\langle g, f\rangle \ .$$

This implies that $f \in D(A^*)$ and therefore

$$0 = \langle g, (A^* - iI)f\rangle$$

for all $g \in D(A)$. Since $D(A)$ is dense, it follows that $f \in \mathrm{Ker}(A^* - iI)$ and hence $f = 0$. The argument is the same for $\mathrm{Ran}(A - iI)$. Next we show that $\mathrm{Ran}(A \pm iI)$ is closed. For any $f \in D(A)$ we have

$$\|(A + iI)f\|^2 = \|Af\|^2 + \|f\|^2$$

since A is symmetric. Thus,

$$\|(A + iI)f\|^2 \geq \|f\|^2 \ . \tag{1}$$

If $g_n \in \mathrm{Ran}(A + iI)$ is a sequence that converges to g in $\mathcal{H}$ then $g_n = (A + iI)f_n$ for some $f_n \in D(A)$. The inequality (1) now implies that f_n is a Cauchy sequence and hence converges to some element f. Since A is closed we must have that $f \in D(A)$ and $(A+iI)f = g$ and hence $g \in \mathrm{Ran}(A + iI)$. Thus we conclude that $\mathrm{Ran}(A + iI) = \mathcal{H}$. The proof for $\mathrm{Ran}(A - iI) = \mathcal{H}$ is the same. Next, we prove that c) implies a). Since A is symmetric, $A \subset A^*$. It remains to show that $D(A^*) \subset D(A)$. Let $g \in D(A^*)$. Since $\mathrm{Ran}(A + iI) = \mathcal{H}$ there exists $h \in D(A)$ with

$$(A^* + iI)g = (A + iI)h$$

or

$$A^*(g - h) = -i(g - h)$$

since $h \in D(A^*)$. Thus, $g - h \in \mathrm{Ker}(A^* + iI)$. Since $\mathrm{Ran}(A - iI) = \mathcal{H}$, $\mathrm{Ker}(A^* + iI) = \{0\}$ and hence $g = h$. Just note that for $f \in \mathrm{Ker}(A^* + iI)$ we have for all $g \in D(A)$

$$0 = \langle g, (A^* + iI)f\rangle = \langle (A - iI)g, f\rangle$$

which implies that $f = 0$ since $\mathrm{Ran}(A - iI) = \mathcal{H}$. □

At first sight it is hard to imagine that the adjoint of a symmetric operator can have an imaginary eigenvalue. Here is an example due to von Neumann. Consider the operator

$$A = \frac{1}{i}\frac{d}{dx}x^3 + x^3\frac{1}{i}\frac{d}{dx}$$

on the domain $D(A) = C_c^\infty(\mathbb{R})$. To be precise for $f \in D(A)$

$$Af(x) = \frac{1}{i}\frac{d}{dx}(x^3 f)(x) + \frac{1}{i}x^3 f'(x)$$

The operator A is symmetric. This is a simple exercise. Consider now the equation

$$\frac{1}{i}\frac{d}{dx}(x^3 f) + x^3\frac{1}{i}\frac{df}{dx} = if \ .$$

Note that f in this equation is not in $D(A)$. So the computation is a formal one. This equation is the same as

$$3x^2 f(x) + 2x^3 f'(x) = -f(x) \ ,$$

a first order linear equation which can be solved explicitly.

$$f'(x) = -(\frac{3}{2x} + \frac{1}{2x^3})f(x)$$

or

$$f(x) = \text{const.}|x|^{-3/2}e^{-\frac{1}{4x^2}} \ .$$

If we set $f(0) = 0$ for $x = 0$, the function is everywhere defined and differentiable, in fact infinitely often differentiable. The function f is in $L^2(\mathbb{R})$ and hence $f \in D(A^*)$. So we have found $f \neq 0, f \in L^2(\mathbb{R})$ such that

$$A^* f = if \ .$$

Reacall that

$$\langle Ag, g\rangle = \langle g, Ag\rangle$$

for all $g \in D(A)$. To understand this a bit better consider

$$\int_{-R}^{R}\left[\frac{1}{i}\frac{d}{dx}(x^3 f) + x^3\frac{1}{i}\frac{df}{dx}\right]\overline{f}dx$$

which, using integration by parts, equals

$$2\frac{1}{i}x^3|f(x)|^2\Big|_{-R}^{R} + \int_{-R}^{R} f\,\overline{\left[\frac{1}{i}\frac{d}{dx}(x^3 f) + x^3\frac{1}{i}\frac{df}{dx}\right]}dx \ .$$

Here R is positive. For our function f we see that

$$2\frac{1}{i}x^3|f(x)|^2\Big|_{-R}^{R} = \text{const.}^2 4\frac{1}{i}e^{-\frac{1}{2R^2}}$$

which does not converge to zero as $R \to \infty$.