1. Bounded operators

Let $\mathcal{H}_1, \mathcal{H}_2$ be two Hilbert spaces with inner product $\langle\cdot,\cdot\rangle_1, \langle\cdot,\cdot\rangle_2$. A linear operator $A : \mathcal{H}_1 \to \mathcal{H}_2$ is bounded if

$$\|A\| := \sup_{0\neq f\in\mathcal{H}_1} \frac{\|Af\|_2}{\|f\|_1} < \infty \ .$$

We denote by $L(\mathcal{H}_1, \langle_2)$ the space of all linear bounded operators mapping $\mathcal{H}_1$ into $\mathcal{H}_2$. It is easy to see that $L(\mathcal{H}_1, \mathcal{H}_2)$ together with the norm $\|A\|$ is a normed linear space.

Theorem 1.1. *The space $L(\mathcal{H}_1, \mathcal{H}_2)$ is a Banach space, i.e., a linear normed complet space.*

Proof. Let A_n be a Cauchy Sequence, i.e., for every $\varepsilon > 0$ there exists N so that

$$\|A_n - A_m\| < \varepsilon$$

whenever $n, m > N$. It follows that for any $f \in \mathcal{H}_1$ the sequence $A_n f$ is a Cauchy sequence. Since $\mathcal{H}_2$ is complete there exists $h \in \mathcal{H}_2$ so that $\lim_{n\to\infty} \|A_n f - h\| = 0$. Since the limit is always unique, this defines an operator A by setting $Af := h$. It is easy to see that A is linear. Since $|\|A_n\| - \|A_m\|| \leq \|A_n - A_m\|$ it follows that $\|A_n\|$ is a Cauchy sequence and hence it is bounded and convergent. Since

$$\|Af\| = \lim_{n\to\infty} \|A_n f\| \leq \lim_{n\to\infty} \|A_n\|\|f\|$$

it follows that A is a bounded operator. Now,

$$\lim_{n\to\infty} \|A - A_n\| = \lim_{n\to\infty} \sup_{\|f\|=1} \|(A - A_n)f\| = \lim_{n\to\infty} \sup_{\|f\|=1} \lim_{m\to\infty} \|(A_m - A_n)f\|$$

$$\leq \lim_{n\to\infty} \sup_{\|f\|=1} \lim_{m\to\infty} \|(A_m - A_n)\|\|f\| = \lim_{n\to\infty} \lim_{m\to\infty} \|(A_m - A_n)\| = 0 \ .$$

□

Definition 1.2. *A linear operator $A : D(A) \to \mathcal{H}$ is invertible if it is onto and ono-to-one.*

Theorem 1.3 (Neumann Series)**.** *Let $T : \mathcal{H} \to \mathcal{H}$ be bounded and assume that $\|T\| < 1$. Then $(I - T)$ is invertible and its inverse is given by the norm convergent Neumann series*

$$(I - T)^{-1} = \sum_{n=0}^{\infty} T^n \ .$$

Proof. The series $\sum_{n=0}^{\infty} T^n$ is norm convergent. For this consider

$$T_N = \sum_{n=0}^{N} T^n$$

and note that for $N > M$

$$\|T_N - T_M\| = \|\sum_{n=M+1}^{N} T^n\| \leq \sum_{n=M+1}^{N} \|T\|^n \leq \frac{\|T\|^{M+1}}{1 - \|T\|}$$

which tends to zero as $M \to \infty$, since $L(\mathcal{H}, \mathcal{H})$ is complete. Now

$$(I - T)T_N = I - T^{N+1}$$

which converges in norm to I. Likewise, $T_N(I - T) = I - T^{N+1}$ also converges to I. Hence $(I - T)^{-1}$ is the inverse of $(I - T)$. □

An operator $A : \mathcal{H}_1 \to \mathcal{H}_2$ is **compact** if every bounded sequence in $\mathcal{H}_1$ is mapped into a sequence in $\mathcal{H}_2$ that has a convergent subsequence.

Compact operators are bounded for assume that A is not bounded. Then there exists a sequence $f_n \in \mathcal{H}_1$ such that $\|f_n\| = 1$ and $\|Af_n\| \geq n$ all $n = 1, 2, \ldots$. This, contradicts thefact that Af_n has a convergent subsequence.

Compact operators have a number of interesting properties and the following theorem is useful for showing that certain operators are compact.

Theorem 1.4. *Let $A_n : \mathcal{H}_1 \to \mathcal{H}_2$ be a sequence of compact operators and assume that there exists A that*

$$\|A - A_n\| \to 0$$

as $n \to \infty$. Then A is compact.

Proof. Let f_i be a bounded sequence. There exists a subsequence $f_{m_1(i)}$ so that $A_1 f_{m_1(i)}$ is convergent. There exists a subsequence $m_2(i)$ of $m_1(i)$ such that $A_2 f_{m_2(i)}$ is convergent. Continuing this way we obtain a subsequence $f_{m_k(i)}$ of $f_{m_{k-1}(i)}$ so that $A_k f_{m_k(i)}$ converges. Passing to the diagonal $f_{m_k(k)}$ we have a sequence so that $A_n f_{m_k(k)}$ converges for all n. Since f_n is bounded there exists C so that $\|f_n\| \leq C$. Now pick $\varepsilon > 0$ arbitrary. Fix n so that $\|A - A_n\| < \varepsilon/(3C)$. Next pick N so that $\|A_n f_{m_k(k)} - A_n f_{m_l(l)}\| < \varepsilon/3$ for all $k, l > N$. Then

$$\|Af_{m_k(k)} - Af_{m_l(l)}\| \leq \|(A - A_n)f_{m_k(k)}\| + \|A_n f_{m_k(k)} - A_n f_{m_l(l)}\| + \|(A_n - A)f_{m_l(l)}\|$$

$$\leq \|(A - A_n)\|\|f_{m_k(k)}\| + \|A_n f_{m_k(k)} - A_n f_{m_l(l)}\| + \|(A_n - A)\|\|f_{m_l(l)}\| < \varepsilon$$

This means that $Af_{m_k(k)}$ is a Cauchy sequence and hence A is compact. □

2. Weak sequences on existence of eigenvalues for compact operators

Below is a short summary about weakly convergent sequences and its uses. Recall that a sequence $f_n \in \mathcal{H}$ converges weakly to $f \in \mathcal{H}$ if

$$\lim_{n \to \infty} (h, f_n) = (h, f)$$

for all elements $h \in \mathcal{H}$. As an example consider on orthonormal sequence e_n which converges weakly to 0. Clearly every strongly convergent sequence is weakly convergent. What the example just mentioned suggests is that there are "many" more sequences that converge weakly than there are that converge strongly.

Among the memorable facts are that any weakly convergent sequence is bounded, i.e., there exists a constant C such that $\|f_n\| \leq C$ for all $n = 1, 2, 3, \ldots$. The point about weak concergence is, however, the following theorem.

Theorem 2.1. Weak sequential compactness *Let $\mathcal{H}$ be a Hilbert space. Then every bounded sequence f_n has a weakly convergent subsequence.*

Proof. A short sketch of the proof: let f_n be a abounded sequence. Then all the possible finite linear combination span a linear manifold $\mathcal{M}$. The closure, i.e., the intersection of all subspaces of $\mathcal{H}$ that contain $\mathcal{M}$ is the closure of $\mathcal{M}$. We denote this space by $\mathcal{G}$. It is a subspace of $\mathcal{H}$, it has a countable dense set of vectors (Why?) and hence forms a separable Hilbert space. We establish now the existence of a convergent subsequence in this Hilbert space. Denote by $N \subset \mathcal{G}$ a countable dense set of vectors. Pick $h_1 \in N$. Since the sequence is f_n is bounded so is (h_1, f_n) and hence there exists a convergent subsequence $(h, f_{n_1(k)})$. Now pick $h_2 \in N$ and a new subsequence $n_2(k)$ of $n_1(k)$ such that $(h_2, f_{n_2(k)})$ converges. Continuing this way we have for any $j = 1, 2, 3, \ldots$

$$\lim_{k\to\infty} (h_j, f_{n_j(k)}) = c_j$$

and $n_j(k)$ is a subsequence of $n_{j-1}(k)$. Now we consider the sequence

$$f_{n_k(k)}$$

and note that

$$\lim_{k\to\infty} (h_j, f_{n_k(k)}) = c_j \ .$$

Sine $f_{n_k(k)}$ is bounded we have that

$$\lim_{k\to\infty} (h, f_{n_k(k)})$$

converges for all $h \in \mathcal{G}$ to some limit $c(h)$. If $u \in \mathcal{H}$ we can write uniquely $u = h + v$ where $h \in \mathcal{G}$ and $v \in \mathcal{G}^\perp$. Since $(v, f_n) = 0$ for all $n = 1, 2, 3, \ldots$ we find that

$$\lim_{k\to infty} (u, f_{n_k(k)}) = \lim_{k\to\infty} (h, f_{n_k(k)}) = c(u) \ .$$

It is easy to see that $u \to c(u)$ is linear and that $|c(u)| \le C\|u\|$. Hence, by the Riesz representation theorem there exists $f \in \mathcal{H}$ such that $c(u) = (u, f)$ for all $u \in \mathcal{H}$. Hence, $f_{n_k(k)}$ converges weakly to f.

□

Let us turn now to compact operators. Recall $A : \mathcal{H}_1 \to \mathcal{H}_2$ is compact if it maps any bounded sequence into a sequence that has a *strongly* convergent subsequence. Alternatively, thanks to Theorem 2.1, as we shall show, $A : \mathcal{H} \to \mathcal{H}$ is compact if and only if it maps weakly convergent sequences into strongly convergent.

Theorem 2.2. *An operator $A : \mathcal{H}_1 \to \mathcal{H}_2$ is compact if and only if it maps weakly convergent sequences into strongly convergent sequences.*

Proof. Assume that A is compact. If f_n is a sequence converging weakly to f, it is bounded by the uniform boundedness prinicple. Assume that Af_n does not converge strongly to Af. There exists $\varepsilon > 0$ and a subsequence $n(k)$ so that

$$\|Af_{n(k)} - Af\| > \varepsilon \ .$$

Since A is compact there exists a further subsequence $n_1(k)$ so that $Af_{n_1(k)}$ converges strongly to some element h. For any $g \in \mathcal{H}_2$ we have

$$\lim_{k\to\infty} \langle Af_{n_1(k)}, g\rangle_2 = \lim_{k\to\infty} \langle f_{n_1(k)}, A^*g\rangle_1 = \langle f, A^*g\rangle_1 = \langle Af, g\rangle_2$$

since f_n converges weakly to f. But

$$\langle h, g\rangle_2 = \lim_{k\to\infty} \langle Af_{n_1(k)}, g\rangle_2$$

and hence $\langle h, g\rangle_2 = \langle Af, g\rangle_2$ for every $g \in \mathcal{H}_2$. Hence $Af = h$ which is a contradiction.

Suppose that A maps weakly convergent sequences into strongly convergent sequences. Let f_n be any bounded sequence. By the weak sequential compactness we can pick a weakly convergetn subsequence again denoted by f_n. Since Af_n converges strongly A is compact. □

3. Sesquilinear forms

Sesquilinear forms are a useful tool for studying linear operators. In what follows we are looking at a Hilbert space $\mathcal{H}$ with inner product denoted by $\langle \cdot, \cdot \rangle$.

Definition 3.1 (Sesquilinear form)**.** *(The word "sesqui" means "one and one half") A sesquilinear form $q(\cdot,\cdot) : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ is linear in the first argument and complex conjugate linear in the second argument.*

If $T : \mathcal{H} \to \mathcal{H}$ is a bounded linear operator, then

$$q_T(f, g) = \langle Tf, g \rangle$$

is a sesquilinear form.

A sesquilinear form is bounded if

$$\sup_{\|f\|=\|g\|=1} |q(f, g)| \leq C$$

where C is some constant. The left side of the above inequality is denoted by$\|q\|$. The sesquilinear form q_T is bounded. Indeed

$$|q_T(f, g)| \leq \|Tf\|\|g\|$$

and recalling that $\|Tf\| \leq \|T\|\|f\|$ we have that

$$\|q_T\| \leq \|T\| \ .$$

On the other hand

$$\|Tf\|^2 = q_T(f, Tf) \leq \|q_T\|\|f\|\|Tf\|$$

and hence

$$\|T\| = \|q_T\|$$

.

A particularly interesting example of sesquilinear forms is furnished by self adjoint operators. If $T^* = T$ then

$$q_T(f, g) = \overline{q_T(g, f)}$$

and in particular

$$|q_T(f, g)| = |q_T(g, f)| \ .$$

Theorem 3.2. *Let q be a bounded sesquilinear form with the additional property that*

$$|q(f, g)| = |q(g, f)| \ .$$

Then

$$\|q\| = \sup_{\|f\|=1} |q(f, f)| \ .$$

An immediate consequence of this inocuous statement is that for any bounded self adjoint operator T

$$\|T\| = \sup_{\|f\|=1} |\langle Tf, f \rangle| \ .$$

Proof of Theorem 3.2. Denote

$$C = \sup_{\|f\|=1} |q(f, f)| \ .$$

It is obvious that $C \le \|q\|$. Pick any $f, g \in \mathcal{H}$ and note that

$$q(f, g) + q(g, f) = \frac{1}{2}[q(f + g, f + g) - q(f - g, f - g)]$$

so that

$$|q(f, g) + q(g, f)| \le \frac{C}{2}[\|f + g\|^2 + \|f - g\|^2] = C[\|f\|^2 + \|g\|^2] \ .$$

We may assume that $q(f, g) \neq 0$ and hence we may define

$$e^{i\theta} = \frac{q(f, g)}{|q(f, g)|} \ , e^{i\psi} = \frac{q(g, f)}{|q(g, f)|}$$

Now

$$\begin{aligned} 2|q(f, g)| &= e^{-i\theta}e^{i\theta}|q(f, g)| + e^{-i\psi}e^{i\psi}|q(g, f)| \\ &= e^{-i\theta}q(f, g) + e^{-i\psi}q(g, f) \\ &= e^{-i\frac{\theta+\psi}{2}}) \left[q(e^{-\frac{i\theta}{2}}f, e^{-\frac{i\psi}{2}}g) + q(e^{-\frac{i\psi}{2}}g, e^{-\frac{i\theta}{2}}f)\right] \\ &\le C\left[\|e^{-\frac{i\theta}{2}}f\|^2 + \|e^{\frac{-i\psi}{2}}g\|^2\right] = C[\|f\|^2 + \|g\|^2] \end{aligned}$$

Hence

$$\sup_{\|f\|=\|g\|=1} |q(f, g)| \le C \ .$$

This proves the theorem.

□

There is a one to one correspondence between linear operators and sesquilinear forms.

Theorem 3.3. *Let q be a bounded sesquilinear form. There exists a unique linear bounded operator $A : \mathcal{H} \to \mathcal{H}$ so that*

$$q(f, g) = q_A(f, g) := \langle Af, g\rangle \ .$$

Proof. The functional $g \to \overline{q(f, g)}$ is a bounded linear functional on $\mathcal{H}$. By Riesz' theorem there exists an element h_f uniquely determined by f so that

$$\overline{q(f, g)} = \langle g, h_f\rangle \ .$$

Hence we can associate for every f a unique h_f. It is easy to see that this map is linear and hence we may define a linear operator $A : \mathcal{H} \to \mathcal{H}$ by

$$Af = h_f \ .$$

Hence

$$q(f, g) = \overline{\langle g, Af\rangle} = \langle Af, g\rangle \ .$$

□

Recall that $f \in \mathcal{H}$ is an eigenvector of the linear operator A if there exists a number $\lambda \in \mathbb{C}$ such that

$$Af = \lambda f \ .$$

Theorem 3.4. *Compact self-adjoint operators have eigenvalues let $A : \mathcal{H} \to \mathcal{H}$ be a compact, self-adjoint operator. Then either $\|A\|$ or $-\|A\|$ is an eigenvalue of A.*

Proof. Here we recall that the sesquilinear form
$$\omega(f,g) := (Af,g)$$
satisfies
$$|\omega(f,g)| = |\omega(g,f)| \ ,$$
and hence
$$\sup_{\|f\|=1} |w(f,f)| = \sup_{\|f\|=1=\|g\|} |\omega(f,g)|$$
and hence
$$\sup_{\|f\|=1} |(Af,f)| = \|A\| \ .$$
We can always assume that
$$\sup_{\|f\|=1} (Af,f) = \|A\|$$
because otherwise we can replace A by $-A$. Our goal is to show that
$$\sup_{\|f\|=1} (Af,f)$$
is attained, i.e., there exists $g \in \mathcal{H}$ with $g \neq 0$ such that
$$Ag = \|A\| g \ .$$
By the definition of the sup there exists a maximizing sequence f_n, that is, $\|f_n\| = 1, n = 1,2,3,\dots$ and
$$\lim_{n\to\infty} (Af_n, f_n) = \|A\| \ .$$
Since f_n is bounded there exists a weakly convergent subsequence which we again denote by f_n whose weak limit we call g. Now, you know from the exercises that
$$1 = \lim_{n\to\infty} \|f_n\| \geq \|g\| \ .$$
Since A is compact Af_n converges strongly to Ag. It is again an exercise to see that
$$\lim_{n\to\infty} (Af_n, f_n) = (Ag,g) \ .$$
In particular $g \neq 0$ since
$$(Ag,g) = \|A\| \ .$$
Next we show that $\|g\| = 1$. Since $0 < \|g\|$,
$$\|A\| \geq (A\frac{g}{\|g\|}, \frac{g}{\|g\|}) = \frac{(Ag,g)}{\|g\|^2} = \frac{\|A\|}{\|g\|^2}$$
and hence $\|g\| \geq 1$ and therefore $\|g\| = 1$. Now we show that
$$Ag = \|A\| g \ .$$
Pick any $v \in \mathcal{H}$ and consider the vector
$$f_t = \frac{g+tv}{\|g+tv\|} \ .$$
The number $t \in \mathbb{C}$. Now
$$\|A\| \geq (Af_t, f_t) = \frac{\|A\| + \bar{t}(Av,g) + t(Ag,v) + |t|^2(Av,v)}{1 + \bar{t}(v,g) + t(g,v) + |t|^2(v,v)}$$

The right side defines a real valued function $F(t)$. Since $F(0) = \|A\|$, f has a maximum at $t = 0$. A short computation shows that for t real

$$0 = \frac{d}{dt}F(0) = (Av, g) + (Ag, v) - \|A\|[(v, g) + (g, v)]$$

and setting $t = is$ we find that

$$0 = \frac{d}{ds}F(is)|_{s=0} = i[-(Av, g) + (Ag, v)] - \|A\|i[-(v, g) + (g, v)] \ .$$

Hence we have that

$$(Ag, v) = \|A\|(g, v)$$

for all $v \in \mathcal{H}$. Thus, $Ag = \|A\|g$. □

4. The spectral theorem for compact operators

We can now use Theorem 3.4 to prove the following theorem.

Theorem 4.1. The Spectral theorem *Let $\mathcal{H}$ be a Hilbert space and $A : \mathcal{H} \to \mathcal{H}$ a linear compact self-adjoint operator. Then there exists an orthonormal system $\varphi_j, j = 1, 2, \dots$ and real numbers $\lambda_j, j = 1, 2, \dots$ with $\lim_{j\to\infty} \lambda_j = 0$ such that for all $f \in \mathcal{H}$*

$$Af = \sum_{j=1}^{\infty} \lambda_j \varphi_j(\varphi_j, f) \ .$$

If $\mathcal{H}$ is separable, the system $\varphi_j, j = 1, 2, \dots$ is a complete orthonormal system.

Proof. By Theorem 3.4 there exists φ_1, normalized such that

$$A\varphi_1 = \pm\|A\|\varphi_1 \ .$$

Note that the minus sign applies if $-\|A\|$ is the the eigenvalue with largest magnitude. Hence we set $\lambda_1 = \pm\|A\|$. The subspace $M_1 = \{f \in \mathcal{H} : (\varphi_1, f) = 0\}$ is a closed subspace of $\mathcal{H}$. Moreover, for $f \in M_1$,

$$(Af, \varphi_1) = (f, A\varphi_1) = \lambda_1(f, \varphi_1) = 0$$

and hence A_1 the restriction of A to the subspace M_1 is a compact self-adjoint operator $A_1 : M_1 \to M_1$. Applying Theorem 3.4 again we obtain a normalized vector φ_2 with

$$A_1\varphi_2 = \pm\|A_1\|\varphi_2 \ .$$

If $\|A_1\| = 0$ then A_1 is the zero operator and we are finished. If not we set $\lambda_2 = \pm\|A_1\|$. Now define the subspace M_2 to be all those vectors in M_1 that are perpendicular to the vector φ_2. The restriction of A_1 to M_2 defines a compact self-adjoint operator A_2. Either it is the zero operator and we are done, or it is not. If not we find a normalized vector φ_3 with $A_2\varphi_3 = \pm\|A_2\|\varphi_3$. Continuing this way we find that either the procedure terminates or there is an infinite sequence pair φ_j, λ_j with

$$A\varphi_j = \lambda_j\varphi_j \ , j = 1, 2, 3, \dots \ .$$

The sequence φ_j is an orthonormal sequence and hence converges weakly to zero. Since A is compact $A\varphi_j$ converges strongly to zero and hence $\lim_{j=1} \lambda_j = 0$. The vectors φ_j span a subspace R of $\mathcal{H}$. If $f \perp R$ then Af must be the zero vector, otherwise we could repeat our procedure, contradicting the fact that $\lim_{j=1} \lambda_j = 0$. Hence R is an invariant subspace of A and A restricted to $R^\perp$ is the zero operator. Thus for $f \in \mathcal{H}$,

$$Af = \sum_{j=1}^{\infty} \lambda_j \varphi_j (\varphi_j, f) \ .$$

Incidentally, this argument shows that the range of a self-adjoint compact operator is always separable. This is generally true for compact operators and not just self-adjoint ones. $\square$