1. The spectral theorem for unitary operators

In this section we give a simple proof of the spectral theorem for unitary operators. We will follow along the same line of thought as we did for bounded self adjoint operators. The main reason for doing this is, because it yields a straightforward proof of the spectral theorem for *unbounded* self adjoint operators. Consider the set $\mathcal{P}$ of all Laurent polynomials, i.e., expressions of the form

$$p(z) = \sum_k c_k z^k \ , z \in \mathbb{C}$$

where the summation range is finite but may contain positive as well as negative integers. Recall that the spectrum of a unitary is a subset of the unit circle. We start with a bound.

Lemma 1.1. *Let V be a unitary operator, let p be a Laurent polynomial and consider*

$$p(V) = \sum_k c_k V^k \ .$$

Then

$$\|p(V)\| = \sup_{z \in \sigma(V)} |p(z)| \ . \tag{1}$$

Proof. Any Laurent polynomial $p(z)$ can be written

$$z^{-N} q(z)$$

where N is an integer and $q(z)$ is a polynomial. Hence

$$p(V) = V^{-N} q(V) \ .$$

We know from the lecture that

$$q(\sigma(V)) = \sigma(q(V)) \ ,$$

and we also know that the spectral radius

$$r(q(V)) = \|q(V)\| \ .$$

Hence we have that

$$\|p(V)\| = \|q(V)\| = r(q(V)) = \sup_{z \in \sigma(q(V))} |z| = \sup_{z \in \sigma(V)} |q(z)| \ .$$

Since the spectrum of a unitary operator is a subset of the unit circle, we find that

$$\sup_{z \in \sigma(V)} |q(z)| = \sup_{z \in \sigma(V)} |z^{-N} q(z)| = \sup_{z \in \sigma(V)} |p(z)| \ .$$

□

The set of Laurent polynomials is obviously an algebra but not closed under complex conjugation. If z is on the unit circle, i.e., of the form $z = e^{i\phi}$ then

$$\overline{p(e^{i\phi})} = \sum_k \overline{c_k} e^{-ik\phi} = \sum_k \overline{c_{-k}} e^{ik\phi} \ .$$

Thus, if p is a Laurent polynomial with coefficients $\{c_k\}$ then we *define* $\overline{p}$ to be the Laurent polynomial with coefficients $\{\overline{c_{-k}}\}$, i.e.,

$$\overline{p}(z) = \sum_k \overline{c_{-k}} z^k \ .$$

In this way we see that the set of all Laurent polynomials, is an algebra that is closed under the operation $p \to \overline{p}$.

Lemma 1.2. *The set of all operators of the form $p(V)$ where $p \in \mathcal{P}$ form a commutative algebra that is closed under *, in fact*

$$(p(V))^* = \overline{p}(V) \ . \tag{2}$$

Hence, its closure is a C^* algebra which we denote by $\mathcal{P}_V$.

Proof. Observing that

$$p(V)^* = \left(\sum_k c_k V^k\right)^* = \sum_k \overline{c_k} V^{*k} = \sum_k \overline{c_{-k}} V^k = \overline{p}(V) \ ,$$

we see that the set is closed under the * operation. The set is clearly commutative and it is elementary to verify that the set is an algebra. □

The Laurnt polynomials in $\mathcal{P}$ when restricted to the unique circle yield all trigonometric polynomials. By Weierstrass's theorem, the trigonometric polynomials on the set $\sigma(V)$ (which is a compact set) are dense in $C(\sigma(V))$, which together with complex conjugation is a C^* algebra. Likewise, by taking the closure in the operator norm of the operators of the form $p(V)$, $p \in \mathcal{P}$ we obtain a C^* algebra $\mathcal{P}_V$. The map that associates with each element $p \in \mathcal{P}$ the operator $p(V)$ is continuous, thanks to Lemma 1.1 and therefore extends uniquely to a map $\Psi : C(\sigma(V)) \to \mathcal{P}_V$.

Lemma 1.3. *The map Ψ is an isometric C^* homorphism. In fact we have for all $p \in C(\sigma(V))$*

$$\|p(V)\| = \sup_{z \in \sigma(V)} |p(z)| \ .$$

Proof. It is straightforward to see that Ψ is linear and and that $\Psi(pq) = \Psi(p)\Psi(q)$. It follows from (2) that for any $p \in C(\sigma(V))$

$$\Psi(p)^* = \Psi(\overline{p}) \ .$$

The isometry follows from Lemma 1.1. □

If p restricted to $\sigma(V)$ is identically zero then $p(V))$ is the zero operator by (1). Hence Ψ induces a map

$$\Phi : C(\sigma(V))/\mathrm{Ker}\Psi \to \mathcal{P}_V$$

which is an isometric C^* algebra isomorphism.

Next pick any vector f, $\|f\| = 1$ and consider the subspace of $\mathcal{H}$ generated by vectors of the form $\Phi(p)f$, $p \in C(\sigma(V))$. This subspace might be dense in $\mathcal{H}$ or not. In any case the closure of this set is a subspace which we denote by $\mathcal{H}_f$. Next, consider the functional

$$p \to \langle f, \Phi(p)f \rangle \ .$$

The functional is linear. It is positive, for if $p \in C(\sigma(V))$ is positive then $p = \overline{g}g$ and hence

$$\langle f, \Phi(p)f \rangle = \langle f, \Phi(\overline{g})\Phi(g)f \rangle = \langle f, \Phi(g)^*\Phi(g)f \rangle = \|\Phi(g)f\|^2 \geq 0 \ .$$

By the Riesz representation theorem there exists a measure μ_f such that

$$\langle f, \Phi(p)f \rangle = \int_{\{\phi : e^{i\phi} \in \sigma(V)\}} p(e^{i\phi}) \mu_f(d\phi).$$

Note that

$$1 = \|f\|^1 = \langle f, \Phi(I) f\rangle = \int_{\{\phi : e^{i\phi} \in \sigma(V)\}} \mu_f(d\phi) \ ,$$

and hence μ is a probability measure. Now we define the operator

$$U : C(\sigma(V)) \to \mathcal{H}_f$$

by setting

$$Up = \Phi(p) f \ .$$

The range of this operator is dense in $\mathcal{H}_f$ and

$$\|Up\|^2 = \|\Phi(p)f\|^2 = \langle f, \Phi(p)^* \Phi(p) f\rangle = \langle f, \Phi(|p|^2)\rangle = \int_{\{\phi : e^{i\phi} \in \sigma(V)\}} |p(e^{i\phi})|^2 \mu_f(d\phi)$$

Since $C(\sigma(V))$ is dense in $L^2(\sigma(V), \mu_f))$ and since Φ is an isomorphism, the operator U extends uniquely to a unitary operator

$$U : L^2(\sigma(V), \mu_f)) \to \mathcal{H}_f \ .$$

For any element $q \in L^2(\sigma(V), \mu_f))$ we find

$$U^{-1} V U q = U^{-1} V q(V) = z q(z) \ , \ z = e^{i\phi}$$

and hence V restricted to $\mathcal{H}_f$ is unitarily equivalent to a multiplication operator. Let $g \perp H_f$ be another vector. Then for any $h \in \mathcal{H}_f$

$$\langle Vg, h\rangle = \langle g, V^* h\rangle = \langle g, V^{-1} h\rangle = 0$$

since $V^{-1} h \in \mathcal{H}_f$. Hence $\mathcal{H}_g \perp \mathcal{H}_f$ and a simple argument, using Zorn's lemma, shows that $\mathcal{H}$ is the orthogonal sum of Hilbert spaces, each of them invariant under V and on each of them V is unitarily equivalent to a multiplication operator. More precisely, the exists an index set I and $f_\alpha \in \mathcal{H}$ so that

$$U : \oplus_{\alpha \in I} L^2(\sigma(V), \mu_{f_\alpha}) \to \mathcal{H}$$

is unitary. Moreover, $U^{-1} V U$ leaves each space $L^2(\sigma(V), \mu_{f_\alpha})$ invariant and restricted to this space $U^{-1} V U p = e^{i\phi} p$.

From this it is not hard to recover the spectral theorem for unbounded self adjoint operators. Assume that $A : D(A) \to \mathcal{H}$ is self adjoint. We know that

$$\mathrm{Ran}(A \pm iI) = \mathcal{H}$$

Pick any $f \in \mathrm{Ran}(A + iI)$. There exists a unique $h \in D(A)$ so that

$$f = (A + iI) h \ .$$

Define

$$Vf = (A - iI) h \ .$$

The operator

$$V : \mathrm{Ran}(A + iI) \to \mathrm{Ran}(A - iI)$$

is onto and

$$\|Vf\|^2 = \|(A + iI)h\|^2 = \|Ah\|^2 + \|h\|^2 = \|f\|^2 \ .$$

Hence, V is unitary. V is called the **Cayley transform** of A. We can recover A from V. If we set

$$h = \frac{1}{2i}(f - Vf) \tag{3}$$

then

$$Ah = \frac{1}{2}(f + Vf) \ . \tag{4}$$

Note that the domain of A consists precisely of those vectors h that are of the form (3).

Theorem 1.4. *Let V be a unitary operator on $\mathcal{H}$ and assume that the set of vectors D of the form*

$$h = \frac{1}{2i}(f - Vf) \ , f \in \mathcal{H} \tag{5}$$

is dense. For any $h \in D$ define

$$Ah = \frac{1}{2}(f + Vf) \ .$$

Then A is self adjoint.

Proof. Note that if f_1, f_2 are two elements with

$$f_1 - Vf_1 = f_2 - Vf_2$$

then with $g = f_1 - f_2$, $Vg = g$. If $h \in D$ then

$$\langle h, g \rangle = \frac{1}{2i}\langle f - Vf, g \rangle = \frac{1}{2i}\langle f, g - V^{-1}g \rangle = 0$$

and, since D is by assumption dense, $g = 0$. This shows that the operator A is well defined, because for any $h \in D$ the f in (5) is *unique*. That A is symmetric is a straightforward computation. Next, A is closed. If $v_n \in D$ with $v_n \to v$ and $Av_n \to u$ then

$$v_n = \frac{1}{2i}(f_n - Vf_n)$$

and

$$Av_n = \frac{1}{2i}(f_n + Vf_n)$$

so that

$$f_n = (A + iI)v_n \ , Vf_n = (A - iI)v_n \ .$$

Hence f_n converges to $f = u + iv$ and Vf_n converges to $u - iv$. Because V is continuous

$$Vf = u - iv$$

and therefore

$$v = \frac{1}{2i}(f - Vf) \in D$$

and

$$u = \frac{1}{2}(f + Vf) = Av \ .$$

Since A is symmetric and closed $Ran(A \pm iI)$ is also closed. Let $g \perp \mathrm{Ran}(A + iI)$. Then

$$\langle (A + iI)h, g \rangle = 0$$

for all $h \in D$. Because

$$h = \frac{1}{2i}(f - Vf)$$

and $Ah = \frac{1}{2}(f + Vf)$, we have that $(A + iI)h = f$ and hence

$$\langle f, g \rangle = 0$$

for all $f \in \mathcal{H}$. Hence $g = 0$ and $\mathrm{Ran}(A + iI) = \mathcal{H}$. The argument for $\mathrm{Ran}(A - iI) = \mathcal{H}$ is similar. Thus, by the fundamental theorem about self adjoint operators A is self adjoint. □

Now, we can use the spectral theorem for unitary operators to prove the spectral theorem for unbounded self adjoint operators.

Theorem 1.5. *Let $A : D(A) \to \mathcal{H}$ be a self adjoint operator. There exists a unitary operator U and a collection of spaces $L^2(\sigma(A), \nu_\alpha)$ such that*

$$U : \oplus_\alpha L^2(\sigma(A), \nu_\alpha) \to \mathcal{H}$$

is unitary. The element $Up, p \in L^2(\sigma(A), \nu_\alpha)$ is in the domain of A if and only if

$$\int_{\sigma(A)} |\lambda|^2 |p(\lambda)|^2 \nu_\alpha(\lambda) < \infty$$

in which case

$$U^{-1}AUp(\lambda) = \lambda p(\lambda) \ .$$

Proof. Let V be the Cayley Transform of A. Consider $f \in D(A)$ and consider the space $\mathcal{H}_f$ for which f is a V-cyclic vector i.e., the span of $\{V^k f\}_{k=-\infty}^{\infty}$ is dense in $\mathcal{H}_f$. V restricted to $\mathcal{H}_f$ is a unitary operator V_f. The set D_f of all vectors h of the form

$$h = \frac{1}{2i}(g - V_f g) \ , \ g \in \mathcal{H}_f$$

is dense in $\mathcal{H}_f$. To see this suppose that there exists $u \in \mathcal{H}_f$ so that

$$\langle \frac{1}{2i}(g - V_f g), u \rangle = 0$$

for all $g \in \mathcal{H}_f$. Then

$$\langle g, u - V_f^{-1} u \rangle = 0$$

for all $g \in \mathcal{H}_f$ and since $u - V_f^{-1}u \in \mathcal{H}_f$ we must have that $V_f u = u$ and hence $Vu = u$. This would imply that the set of all vectors of the form

$$\frac{1}{2i}(g - Vg)$$

is not dense in $\mathcal{H}$, a contradiction. By the previous theorem, the operator V_f defines a self adjoint operator B on $\mathcal{H}_f$ which is easily seen to be A restricted to D_f. Since we shall be working exclusively in the space $\mathcal{H}_f$ we shall drop the subscript which is the same as assuming that the f is a cyclic vector for V.

Pick any function $g \in \mathcal{H}$. By the spectral theorem for unitary operators we can write $g = Up_g$ for some function $p_g \in L^(\sigma(V), \mu)$ and hence

$$U^{-1}h(e^{i\phi}) = \frac{1}{2i}(p_g - U^{-1}VUp_g)(e^{i\phi}) = \frac{1}{2i}(1 - e^{i\phi})p_g(e^{i\phi})$$

and

$$U^{-1}Ah(e^{i\phi}) = \frac{1}{2}(p_g + U^{-1}VUp_g)(e^{i\phi}) = \frac{1}{2}(1 + e^{i\phi})p_g(e^{i\phi}) \ .$$

It follows that

$$U^{-1}Ah(e^{i\phi}) = \text{cotan}(\phi/2) U^{-1}h(e^{i\phi}) \ ,$$

or

$$U^{-1}AU = M_{\text{cotan}(\phi/2)} \ .$$

Hence, $h \in D(A)$ if and only if $p_g = U^{-1}h$ satisfies

$$\int_{\{\phi : e^{i\phi} \in \sigma(V)\}} [\text{cotan}(\phi/2)]^2 |p_g|^2 \mu(d\phi) < \infty \ .$$

If $h \in D(A)$ then

$$Ah = i(V+1)(V-1)^{-1}h$$

and the spectrum of A is the image of $\sigma(V)$ by the function

$$i\frac{z+1}{z-1}$$

which maps the unit circle to the real line.

Pick any set $S \subset \sigma(A)$ and consider the 'push - forward' of the measure μ onto $\sigma(A)$ given by

$$\nu(S) = \int_{\{\phi:\text{cotan}(\phi/2)\in S\}} \mu(d\phi) \ .$$

Then

$$\int_{\{\phi:e^{i\phi}\in\sigma(V)\}} [\text{cotan}(\phi/2)]^2|p_g|^2\mu(d\phi) = \int_{\sigma(A)} \lambda^2|p_g(\text{cotan}^{-1}(\lambda)|^2\nu(d\lambda) \ .$$

and

$$U^{-1}AU = M_\lambda$$

where M_λ is multiplication by λ. □

The spectral theorem is actually useful. E.g., consider the initial value problem

$$\frac{df}{dt} = -iAf \ , \ f|_{t=0} = f_0 \tag{6}$$

where A is a self adjoint operator. Assume that $f_0 \in D(A)$. Using the spectral theorem this equation is equivalent to the equation

$$\frac{dp}{dt}(\lambda) = -i\lambda p(\lambda)$$

where $p = U^{-1}f$. This equation is readily solved by

$$p(\lambda, t) = e^{-i\lambda t}p_0(\lambda)$$

where

$$p_0(\lambda) = U^{-1}f_0 \ .$$

Therefore, the solution is given by

$$f = Ue^{-iM_\lambda t}U^{-1}f_0$$

and we have established a global existence result for the differential equation (6). Note that, because the domain $D(A)$ is dense, the time evolution is in fact defined for all $f_0 \in \mathcal{H}$. Also note that the operator $Ue^{-iM_\lambda t}U^{-1}$ is in fact a unitary operator.