1. Application of functional analysis to PDEs

1.1. **Introduction.** In this section we give a little introduction to partial differential equations. In particular we consider the problem

$$-\Delta u(x) = f(x) \ x \in \Omega \ , \ u(x) = 0 \ x \in \partial\Omega \tag{1}$$

where Ω is some open bounded domain in $\mathbb{R}^n$. The condition on u on the boundary is called a 'Dirichlet' boundary condition. We shall assume that $f \in L^2_|Omega)$.

This equation can be solved explicitely for a limited number of situations where the domain has symmetry, like a ball or a half space. In general to infer the existence of a solution is a difficult problem.

In order to harness the power of functional analysis for proving the existence of a solution one *relaxes* the problem.

Recall the definition of $H^1(\Omega)$ form the exercises. These were all functions $f \in L^2(\Omega$ with the property that there exist function $g^i_f \in L^2(\Omega)$ so that

$$\int_\Omega f \frac{\partial \phi}{\partial x_i} dx = - \int_\Omega g^i_f \phi dx$$

for all $\phi \in C^\infty_c(\Omega)$. It was shown in the exercises that $H^1(\Omega)$ endowed with the inner product

$$\int_\Omega f \overline{g} dx + \sum_{i=1}^n \int_\Omega \overline{g^i_f} g^i_h dx \tag{2}$$

is a Hilbert space. We shall, henceforth, abuse notation and write

$$g^i_f(x) = \frac{\partial f}{\partial x_i}(x) \ .$$

Note that since Ω is bounded the constant function is certainly in $H^1(\Omega)$. What about functions that vanish on the boundary. Note that anay function $\psi \in C^\infty_c(\Omega)$ is in $H^1(\Omega)$. in fact, the eake derivative equals the usual derivative in this case (why'?).

To capture functions in $H^1(\Omega)$ that vanish on the boundary of Ω we introduce a new space.

Definition 1.1. *The space $H^1_0(\Omega)$ is the closure of the set $C^\infty_c(\Omega)$ in the norm of $H^1(\Omega)$. It is a Hilbert space with the inner product* (2)*.*

Almost by definition this space is a Hilbert space. If f_n is a Cauchy sequence in $H^1_0(\Omega)$ then it converges in $H^1(\Omega)$ to some element f. Since each $f_n \in H^1_0(\Omega)$ there exists $\phi_n \in C^\infty_c(\Omega)$ so that

$$\|f_n - \phi_n\|_{H^1(\Omega)} < \frac{1}{n} \ .$$

Hence

$$\|f - \phi_n\|_{H^1(\Omega)} \le \|f - f_n\|_{H^1(\Omega)} + \|f_n - \phi_n\|_{H^1(\Omega)} < \|f - f_n\|_{H^1(\Omega)} + \frac{1}{n}$$

which tends to zero as $n \to \infty$. Hence $f \in H^1_0(\Omega)$.

Definition 1.2. *$u \in H^1_0(\Omega)$ is a* **weak** *solution of* (1) *if for every $v \in H^1_0(\Omega)$ we have that*

$$\int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega f v dx \tag{3}$$

Note that we have dropped the complex conjugation since we shall be dealing with real valued functions. Also note that the derivatives are all in the weak sense.

Suppose that u is a twice differentiable solution of (1). Then we can integrate by parts and obtain

$$\int_\Omega [-\Delta u - f] v dx = 0$$

from which we conclude that $-\Delta u = f$. The path to this stage is, however, thorny and we shall give some indication how to procedd at the end.

1.2. **An important inequality.** Let f be a smooth function of the intervall $[0, a]$ and assume that $f(a) = f(0) = 0$. Our goal is to get a lower bound on the ratio

$$\frac{\int_0^a |f'(x)|^2 dx}{\int_0^a |f(x)|^2 dx} \ . \tag{4}$$

We know from Fourier analysis that any smooth function can be expanded in a Fourier series of the type

$$f(x) = \sum_{k=1}^{\infty} c_k \sqrt{2} \sin(\frac{\pi k x}{a})$$

The functions

$$\sqrt{2} \sin(\frac{\pi k x}{a})$$

form an orthonormal system and hence we have that

$$\int_0^a |f(x)|^2 dx = \sum_{k=1}^{\infty} |c_k|^2 \ .$$

The function f' is given by

$$f'(x) = \frac{\pi}{a} \sum_{k=1}^{\infty} k c_k \sqrt{2} \cos(\frac{\pi k x}{a})$$

and once more using orthogonality we find that

$$\int_0^a |f'(x)|^2 dx = \left(\frac{\pi}{a}\right)^2 \sum_{k=1}^{\infty} k^2 |c_k|^2 \ .$$

Hence our ratio (4) can be expressed as

$$\left(\frac{\pi}{a}\right)^2 \frac{\sum_{k=1}^\infty k^2 |c_k|^2}{\sum_{k=1}^\infty |c_k|^2} \ .$$

Clearly by replacing k^2 by its smalles value 1 we find that

$$\left(\frac{\pi}{a}\right)^2 \frac{\sum_{k=1}^\infty k^2 |c_k|^2}{\sum_{k=1}^\infty |c_k|^2} \geq \left(\frac{\pi}{a}\right)^2$$

with equality if and only if $c_k = 0$ for all $k > 1$. Thus, we proved

Theorem 1.3 (Wirtinger's inequality)**.** *For any smooth function f on the interval $[0,a]$ with $f(0) = f(a) = 0$ we have that*

$$\int_0^a |f'(x)|^2 dx \geq \int_0^a |f(x)|^2 dx$$

with equality only if f is a multiple of $\sin(\frac{\pi x}{a})$*.*

Here is another maybe even simpler proof for this inequality. Since $f(0) = f(a) = 0$ we may write

$$f(x) = h(x)g(x)$$

where $h(x) = \sin(\frac{\pi x}{a})$ and $g(x)$ some function which has compact support. Now

$$f'(x) = h'(x)g(x) + h(x)g'(x)$$

so that

$$\int_0^a |f'(x)|^2 dx = \int_0^a h'(x)^2 g(x)^2 dx + \int_0^a h(x)^2 g'(x)^2 dx + \int_0^a h(x)h'(x)[g(x)^2]' dx \ .$$

Integrating the last term by parts and noting that the boundary terms vanish we get

$$\int_0^a |f'(x)|^2 dx = \int_0^a h'(x)^2 g(x)^2 dx + \int_0^a h(x)^2 g'(x)^2 dx - \int_0^a [h(x)h'(x)]'[g(x)^2] dx \ .$$

$$= \int_0^a h(x)^2 g'(x)^2 dx - \int_0^a h(x)h''(x)[g(x)^2] dx \ .$$

Since

$$-h''(x) = \left(\frac{\pi}{a}\right)^2 h(x)$$

we find

$$\int_0^a |f'(x)|^2 dx = \int_0^a h(x)^2 g'(x)^2 dx + \left(\frac{\pi}{a}\right)^2 \int_0^a h(x)^2 [g(x)^2] dx \geq \left(\frac{\pi}{a}\right)^2 \int_0^a f(x)^2 dx \ .$$

This theorem has an interesting consequence. Let $\Omega \in \mathbb{R}^n$ be an open set and assume that Ω fits into a slab, i.e., between two parallel $n-1$ dimensional planes. Denote by D the infimum of the distances of the pairs of planes so that Ω fits between them.

Theorem 1.4. *Let $u \in C_c^\infty(\Omega)$. Then*

$$\int_\Omega |\nabla u(x)|^2 dx \geq \left(\frac{\pi}{D}\right)^2 \int_\Omega |u(x)|^2 dx \ .$$

Proof. let $u \in C_c^\infty(\Omega)$. The support C, i.e., the closure of the set where the function does not vanish is a compact set, by assumption. Hence it has a diameter $d < D$ (why?). Pick any $0 < \delta < D - d$ There exist two planes a distance $d + \delta$ appart so that C fits between these two planes (why?). By rotating the function we can assume that these two planes are perpendicular to the x-axis and by translating the function w can assum that one of the planes passes through the origin and the other through the point $(d + \delta, 0, \dots, 0)$. We estimate

$$\int_\Omega |\nabla u|^2 dx = \int_{\mathbb{R}^{n-1}} \int_0^{d+\delta} |\nabla u|^2 dx \geq \int_{\mathbb{R}^{n-1}} \left[\int_0^{d+\delta} |\frac{\partial u}{\partial x_1}|^2 dx_1 \right] dx_2 \cdots dx_n$$

Using Wirtinger's inequality we find for every fixed $x_2, \dots, x_n$

$$\int_0^{d+\delta} |\frac{\partial u}{\partial x_1}|^2 dx_1 \geq \left(\frac{\pi}{d+\delta}\right)^2 \int_0^{d+\delta} |u|^2 dx_1$$

which implies the result. $\square$

As an immediate corollary we have

Theorem 1.5. *Let u be any function in $H_0^1(\Omega)$. Then*

$$\int_\Omega |\nabla u(x)|^2 dx \geq \left(\frac{\pi}{D}\right)^2 \int_\Omega |u(x)|^2 dx \ .$$

In particular

$$\sqrt{\int_\Omega |\nabla u(x)|^2 dx}$$

is a norm which is equivalent with $\|u\|_{H^1(\Omega)}$ and $H_0^1(\Omega)$ with inner product

$$(u,v)_0 = \int_\Omega \nabla u \cdot \nabla v dx$$

is a Hilbert space.

Proof. Let $u \in H_0^1(\Omega)$. Then there exists a sequence of function $\phi_n \in C_c^\infty(\Omega)$ so that $\|u - \phi_n\|_{H^1(\Omega)} \to 0$ as $n \to \infty$. Hence

$$\int_\Omega |\nabla u - \nabla \phi|^2 dx \to 0$$

and

$$\int_\Omega |u - \phi_n|^2 dx \to 0$$

as $n \to \infty$. Hence

$$\int_\Omega |\nabla u|^2 dx = \lim_{n\to\infty} \int_| \nabla \phi_n|^2 dx \geq \left(\frac{\pi}{D}\right)^2 \int \lim_{n\to\infty} \int_\Omega |\phi_n(x)|^2 dx = \left(\frac{\pi}{D}\right)^2 \int_\Omega |u(x)|^2 dx \ .$$

Further

$$\int_\Omega |\nabla u|^2 dx \leq \|u\|^2_{H^1(\Omega)} = \int_\Omega |u(x)|^2 dx + \int_\Omega |\nabla u|^2 dx \leq \left(\left(\frac{D}{\pi}\right)^2 + 1\right) \int_\Omega |\nabla u|^2 dx$$

and hence the norms are equivalent. The final statement is immediate. $\square$

1.3. **Existence of a weak solution.**

Theorem 1.6. *There exists a unique function $u \in H_0^1(\Omega)$ that satisfies*

$$\int_\Omega f v dx = \int_\Omega \nabla u \cdot \nabla v dx$$

for all $v \in H_0^1(\Omega)$.

Proof. This will fall out from the Riesz representation theorem. Consider the linear functional

$$v \to \int_\Omega f v dx$$

and note that it is bounded on $H_0^1(\Omega)$ since

$$|\int_\Omega f v dx| \leq (\int_\Omega |f|^2 dx)^{1/2} (\int_\Omega |v|^2 dx)^{1/2}$$

and since $v \in H^1(\Omega)$ we have that

$$(\int_\Omega |v|^2 dx)^{1/2} \le \frac{\pi}{D}(\int_\Omega |\nabla v|^2 dx)^{1/2} \ .$$

By the Riesz representation theorem there exists a unique $u \in H^1_0(\Omega)$ so that

$$\int_\Omega f v dx = \int_\Omega \nabla u \cdot \nabla v dx$$

which proves the existence of the weak solution. □

Now the hard work starts. It would be nice to be able to integrate by parts and to obtain

$$\int_\Omega [-\Delta u - f] v dx = 0$$

for all $v \in H^1_0(\Omega)$. At least we could conclude that $-\Delta u = f$ pointwise almost everywhere in Ω Note, however, that the 'solution' u is in $H^1_0(\Omega)$ only and we have no clue in what sense one should interpret the second derivative of u.

It is also interesting that at this stage one has only (3) to work with and this leads to the **theory of elliptic regularity** which was developped around 1950.

First, in the same way we introduced $H^1(\Omega)$, one introduces the higher Sobolev spaces $H^k(\Omega)$. Then one proves that a function that is in a high enough Soboleve space $H^k(\Omega)$ is in fact continuously differentiable. In particular a function that is in $H^k(\Omega)$ for all k is C^∞ in the interior. One can also prove that a function that is in $H^k(\Omega)$ for all k, is smooth on the boundary provided that the boundary $\partial\Omega$ itself is smooth. This kind of analysis has nothing to do with the PDE per se.

In a first step one proves that the weak solution u is in $H^2(U)$ where is an open subset of Ω whose closure is compact. If $f \in H^k(\Omega)$ one can actually prove that $u \in H^{k+2}(U)$. This step is called **interior regularity**. The way to do this is by carefully chosen test functions.

The next problem is the boundary. One would like to conclude that the function u vanishes on the boundary but this is tricky since any function in a Sobolev space is defined only almost everywhere and since the boundary has measure zero one has to carefully define what one means by 'boundary' value. This kind of theorems are known as **trace theorems**. Now, if the boundary is smooth one proceeds to prove higher regularity of u which, assuming that f and $\partial\Omega$ are smooth, that $u \in C^\infty(\overline{\Omega})$ and the equation (1) holds in the usual sense.