1. The spectraltheorem for self adjoint operators

Again, we follow the book of **R. Zimmer, Essential results of functional analysis** almost verbatim.

The concept of a Banach algebra does not make any mantion of adjoint operators. This is an additional structure whch has to be defined when dealing with this abstract setting. The relevant notion is the one of a C^* algebra.

Definition 1.1. *A C^*-algebra is a Banach algebra $\mathcal{A}$ with the operation $* : \mathcal{A} \to \mathcal{A}$ sending x to x^*. This operation has to satisfy the following conditions.*

a) $x^{**} = x$
b) $(cx)^* = \overline{c}x^*$ *for all* $x \in \mathcal{A}$ *and all* $c \in \mathbb{C}$.
c) $(xy)^* = y^*x^*$ *for all* $x, y \in \mathcal{A}$
d) $\|x^*x\| = \|x\|^2$
e) If $\mathcal{A}$ *has an identity* I*, then* $I^* = I$.

Note that the statement about the idendity follows from c) and a). First we note that the identity is unique. Now from c) we have that $x = (x^*I)^* = I^*x$ and $x = (Ix^*)^* = xI^*$ for all $x \in \mathcal{B}$, hence $I^* = I$. (I tank Alex for this remark).

Examples:

a) The standard example of such an algebra is, of course, given by bounded operators on a Hilbert space. All the properties except for d) are obvious. To see d) note that for any bounded operator T we have that

$$\|T^*T\| = \sup_{\|f\|=\|g\|=1} \langle f, T^*Tg\rangle = \sup_{\|f\|=1} \langle f, T^*Tf\rangle = \sup_{\|f\|=1} \|Tf\|^2 = \|T\|^2 \ .$$

(Why does the second equality hold?)

b) A simple example is furnished by the set of continuous functions on a compact set S. The spacce $C(S)$ is defined as the set of continuous, complex valued functions. For the norm we take

$$\|f\| = \max_{x \in S} |f(x)|$$

and multiplication is defined as

$$(fg)(x) = f(x)g(x) \ .$$

The space $C(S)$ is a Banach space because $C(S)$ is closed under uniform convergence and hence a Banach algebra. If we define

$$f^*(x) = \overline{f(x)}$$

$C(S)$ satisfies all the properties of a C^* algebra, in fact it is a commutative C^* algebra. We may add the function 1 and get a C^* algebra with identity. The properties are all very easy to verify.

c) Let A be a bounded operator on a Hilbert space. If $p(z)$ is a polynomial, then we may consider $p(A)$ and we see that the set

$$\mathcal{P}_A := \{p(A) : p(z) \text{ a polynomial}\}$$

is an algebra. $\mathcal{P}_A$ is not closed but since $\mathcal{P}_A \subset \mathcal{L}(H)$ one can consider the closure $\overline{\mathcal{P}_A}$ which is now a Banach algebra. It is not a C^* algebra since the operator $A^* \notin \overline{\mathcal{P}_A}$. If, however, $A = A^*$ then we can $\mathcal{P}_A$ is a C^* algebra, since

$$p(A)^* = \overline{p}(A)$$

where $\overline{p}(z)$ is the complex conjugate polynomial, i.e., the polynomial with the complex conjugate coefficients of $p(z)$. In this case $\overline{\mathcal{P}_A}$ is a commutative C^* algebra. It is a sub-algebra of $\mathcal{L}(H)$.

The first part of proving the spectral theorem for self adjoint operators consists in identifying the C^* algebra $\overline{\mathcal{P}_A}$ as the space of continuous functions on $\sigma(A)$.

The relevant notion here is the one of an isomorphism between Banach algebras and, more specifically, C^* algebras.

Definition 1.2. *If $\mathcal{B}_1$ and $\mathcal{B}_2$ are two Banach algebras then we call a map $M : \mathcal{B}_1 \to \mathcal{B}_2$ an isomorphism if the following holds.*

a) M is linear.

b) M is an isometry, i.e., $\|M(x)\|_2 = \|x\|_1, \text{ all } x \in \mathcal{B}_1$.

c) M is invertible on $\mathcal{B}_1$.

The properties a) - c) turn M into an isomorphism of Banach spaces. The next property turns M into an isomorphism of Banach algebras.

d) For all $x, y \in \mathcal{B}_1$, $M(xy) = M(x)M(y)$.

If $\mathcal{A}_1, \mathcal{A}_2$ are C^* algebras, then $M : \mathcal{A}_1 \to \mathcal{A}_2$ is an C^* isomorphism if it is an isomorphism between Banach algebras and in addition

$$M(x)^* = M(x^*)$$

for all $x \in \mathcal{A}_1$. If $\mathcal{A}_1$ nad $\mathcal{A}_2$ are C^* algebras with identity, then we alsy require $M(I_1) = I_2$.

In what follows A is a bounded self adjoint operator. The first part for proving the spectral theorem consists in establishing a C^* isomorphism between $\overline{P_A}$ and the space of continuous functions on the spectrum of A.

Theorem 1.3. *There exists a unique C^* algebra isomorphism Φ between $C(\sigma(A))$ the space of continuous functions on the spectrum of A, and $\overline{\mathcal{P}_A}$, the closure of the set*

$$\{p(A) : p(z) \text{ a polynomial}\}$$

in $\mathcal{L}(\mathcal{H})$ with

$$\Phi(p) = p(A)$$

where p is a polynomial considered as a function on $\sigma(A)$.

Thus, from an algebraic perspective there is no distinction between the algebra $\overline{\mathcal{P}_A}$ and the algebra $C(\sigma(A))$. Of course, their actual representation is very different. One is an algebra of operators on a Hilbert space and the other continuous functions on a compact space.

Proof. The proof is relatively easy. Define a map $\Psi : \mathbb{C}[z] \to \mathcal{P}_A$ by

$$\Psi(p) = p(A) \ .$$

Here $\mathbb{C}[z]$ denotes the space pf all polynomials in the variable z. It is obviously an algebra homeomorphism and

$$\overline{p}(A) = p(A)^* \ .$$

Suppose that $p \in \mathbb{C}[z]$ vanishes identically on $\sigma(A)$. We now from Theorem 1.6 in the previous section that $\sigma(p(A)) = p(\sigma(A))$ and hence $\sigma(p(A)) = \{0\}$. This implies that the spectral radius of $p(A)$, $r(p(A)) = 0$ and hence by Corollary 1.12 in the previous section, we have that $\|p(A)\| = 0$ and $p(A)$ is the zero operator. Thus, the space of all polynomials that vanish on

$\sigma(A)$ forms the kernel of Ψ, KerΨ. If $\pi : \mathbb{C}[z] \to \mathbb{C}[z]/\text{Ker}\Psi$ denotes the canonical projection, we know that there exists a unique $\Phi : \mathbb{C}[z]/\text{Ker}\Psi \to \mathcal{P}_A$ such that

$$\Psi = \Phi \circ \pi \ .$$

We denote the space

$$\mathbb{C}[z]/Ker\Psi = P_{\sigma(A)} \ .$$

Any two polynomials in $P_{\sigma(A)}$ are equivalent if their difference is a polynomial that vanishes on $\sigma(A)$. The map Φ is obviously a homomorphism. Further, since

$$\sup\{|p(z)| : z \in \sigma(A)\} = r(p(A)) = \|p(A)\|$$

Φ is an isometry into $\mathcal{P}_A$. By the Stone-Weierstrass Theorem, the set $P_{\sigma(A)}$ is dense in the sup norm in $C(\sigma(A))$ and Φ extends uniquely to an isometric homomorphism

$$\Phi : C(\sigma(A)) \to \overline{\mathcal{P}}_A \ .$$

Since the set of operators of the form $p(A)$ is dense in $\overline{\mathcal{P}}_A$ by definition, the map Φ is an isomorphism.

□

So far the underlying Hilbert space was not part of our considerations.

Definition 1.4. *Let A be a bounded operator on a Hilbert space $\mathcal{H}$. A vector $f \in \mathcal{H}$ is a* **cyclic vector for** *A if the span of the vectors $\{A^n f\}_{n=0}^{\infty}$ is dense in $\mathcal{H}$. Equivalently, f is cyclic if the smallest subspace of $\mathcal{H}$ which contains f and is invariant under A is $\mathcal{H}$. Quite generally we can say that if $\mathcal{B} \subset \mathcal{L}(\mathcal{H})$ is a subalgebra, we say that f is a $\mathcal{B}$ - cyclic vector for $\mathcal{B}$ if the set $\{Cf : C \in \mathcal{B}\}$ is dense in $\mathcal{H}$.*

Note that not every vector is a cyclic vector. E.g., take a self adjoint $n \times n$ matrix A. Then an eigenvector is not a cyclic vector for the the whole space. It is of course a cyclic vector for the one dimensional eigenspace. To get a better feeling for this notion we shall argue that any $n \times n$ matrix A with distinct eigenvalues has a cyclic vector. Since the eigenvalues are distinct there exists a basis of eigenvectors. Thus, it is sufficient to find a cyclic vector for a diagonal matrix D with distinct diagonal elements $\lambda_1, \dots, \lambda_n$. Pick any vector x with none of its entries zero. If this vector is not cyclic, the vectors $x, Dx, \dots, D^{n-1}x$ must be linearly dependent, i.e.,

$$p(D)x = 0$$

where

$$p(z) = \sum_{k=0}^{n-1} c_k z^k \ .$$

Factoring this polynomial yields $p(z) = \Pi_{j=1}^{n-1}(z - \mu_j)$ so that

$$\Pi_{j=1}^{n-1}(D - \mu_j I)x = 0 \ .$$

Since none of the entries of x are zero we find that

$$\Pi_{j=1}^{n-1}(\lambda_k - \mu_j) = 0 \ , \text{for } k = 1, 2, \dots, n \ ,$$

which means that we have n distinct roots for a polynomial of degree $n-1$ which cannot be.

Lemma 1.5. *Let $\mathcal{B} \subset \mathcal{L}(\mathcal{H})$ be a subalgebra with the property that if $C \in \mathcal{B}$ then $C^* \in \mathcal{B}$. Then $\mathcal{H}$ is the orthogonal sum of subspaces $\mathcal{H}_i$ each of which is invariant under $\mathcal{B}$ and each possessing an $\mathcal{B}$ cyclic vector.*

Proof. Consider the set of all subspaces of the form

$$V = \sum^{\oplus} V_i$$

where for each i, V_i is an invariant subspace of $\mathcal{B}$ and possesses a $\mathcal{B}$-cyclic vector. We give this set a partial ordering by saying that $V > W$ if $\{W_j : j \in J\} \subset \{V_i : i \in I\}$. This set of subspaces is not empty since we can pick any vector f and consider $\overline{\{\mathcal{B}f\}}$ which is an invariant subspace which possesses obviously f as a $\mathcal{B}$-cyclic vector. Now consider a chain $V_\alpha, \alpha \in I$ where I is an index set. To be a chain means that if $\alpha_1, \alpha_2 \in I$ and

$$V_{\alpha_1} = \sum_{i \in I_{\alpha_1}}^{\oplus} V_{\alpha_1, i}$$

$$V_{\alpha_2} = \sum_{i \in I_{\alpha_2}}^{\oplus} V_{\alpha_2, i}$$

we have that $\{V_{\alpha_1,i} : i \in I_{\alpha_1}\} \subset \{V_{\alpha_2,i} : i \in I_{\alpha_2}\}$ or $\{V_{\alpha_2,i} : i \in I_{\alpha_2}\} \subset \{V_{\alpha_1,i} : i \in I_{\alpha_1}\}$. Such a chain has an upper bound, just take $\cup_{\alpha \in I}\{V_{\alpha,i} : i \in I_\alpha\}$. By Zorn's lemma there exists a maximal element which we denote as

$$V = \sum^{\oplus} V_i \ .$$

We show that $V = \mathcal{H}$. Pick any $f \perp V$. For any $C \in \mathcal{B}$ and $g \in V$ we have that

$$\langle Cf, g\rangle = \langle f, C^* g\rangle$$

and since $C^* \in \mathcal{B}$ we have that $C^* g \in V$ and hence $Cf \perp V$. Thus the space $W = \overline{\{\mathcal{B}f\}}$ is an invariant subsapce and f is a $\mathcal{B}$ - cyclic vector in this space moreover it is rothogonal to V and

$$V \oplus W > V$$

which contradicts the maximality of V. Hence $V = \mathcal{H}$. □

This Lemma reduces to the problem of proving the spectral theorem for a self adjoint operator with a cyclic vector. We shall need the isomorphism constructed in Theorem 1.3 .

Theorem 1.6. *Let $\mathcal{A}$ be a commutative C^* subalgebra of $\mathcal{L}(\mathcal{H})$ with $I \in \mathcal{A}$ and let f be an $\mathcal{A}$ - cyclic vector in $\mathcal{H}$. Assume that there exist an isomorphism $\Phi : \mathcal{A} \to C(X)$ where X is a compact subset of $\mathbb{C}$. There exists a measure μ on X and a unitary operator $U : L^2(X.\mu) \to \mathcal{H}$ so that for all $g \in L^2(X, \mu)$*

$$U^{-1} A U g = \Phi(A) g \ ,$$

for every $A \in \mathcal{A}$.

Proof. Let $f \in \mathcal{H}$, $\|f\| = 1$ be an $\mathcal{A}$-cyclic vector and let $p \in C(X)$. (Note that p is a continuous function and not just a polynomial.) Consider the operator $\Phi^{-1}(p) \in \mathcal{A}$ and consider the functional

$$\ell(p) = \langle \Phi^{-1}(p) f, f\rangle \ .$$

The functional $\mu : C(X) \to \mathbb{C}$ is linear and

$$|\ell(p)| \le \|\Phi^{-1}(p)\| = \|p\|$$

where we recall that

$$\|p\| = \max_X |p(z)|$$

and that Φ is an isometry. Hence ℓ is a bounded linear function on $C(X)$ with $\|\ell\| \leq 1$. Since $\Phi^{-1}(1) = I$ we have In fact $\ell(1) = \langle f, f \rangle = 1$ and hence $\|\ell\| = 1$. The functional ℓ is also a positive functional. In fact if $p \geq 0$ we can write $p = \overline{q}q$ and find

$$\ell(p) = \langle \Phi^{-1}(\overline{q}q)f, f \rangle = \langle \Phi^{-1}(\overline{q})\Phi^{-1}(q)f, f \rangle$$

and since $\Phi(\overline{q}) = \Phi^{-1*}(q)$ we have that

$$\ell(p) = \langle \Phi^{-1}(q)f, \Phi^{-1}(q)f \rangle \geq 0 \ .$$

By the Riesz representation theorem, there exists a measure μ in X such that

$$\ell(p) = \int_X p(z)\mu(dz) \ .$$

Since

$$1 = \ell(1) = \int_X \mu(dz)$$

this measure μ is in fact a probability measure. Thus,

$$\langle \Phi^{-1}(p)f, f \rangle = \int p(z)\mu(dz) \ .$$

Now, we proceed in a strightforward fashion. Define the linear operator

$$U : C(X) \to \mathcal{H}$$

by setting

$$Up = \Phi^{-1}(p)f \ ,$$

and compute

$$\langle Up, Up \rangle = \langle \Phi^{-1}(p)f, \Phi^{-1}(p)f \rangle = \langle \Phi^{-1*}(p)\Phi^{-1}(p)f, f \rangle$$

$$= \langle \Phi^{-1}(\overline{p}p)f, f \rangle = \int_X |p(z)|^2 \mu(dz) \ .$$

Hence if we consider $C(X)$ as a subspace of $L^2(X, \mu)$ we find that U is an isometry. Further the set of all vectors of the form Up, $p \in C(X)$ is the same as the set of all vector of the form Af, $A \in \mathcal{A}$. Hence the range of U is dense in $\mathcal{H}$ and hence U extends uniquely to a unitary isomorphism to all of $L^2(X, \mu)$. For $p, g \in C(X)$

$$U^{-1}\Phi^{-1}(p)Ug = U^{-1}\Phi^{-1}(p)\Phi^{-1}(g)f = U^{-1}\Phi^{-1}(pg)f = M_p g \ ,$$

where M_p is the operator by multiplication with p. In other words, if $A \in \mathcal{A}$ is an operator, we can write $A = \Phi^{-1}(p)$, i.e., $p = \Phi(A) \in C(X)$, and hence the operator A is unitarily equivalent to multiplication by $\Phi(A)$ on $L^2(X, \mu)$. □

We can combine Lemma 1.5 and Theorem 1.6 and prove

Corollary 1.7. *Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$. Then A is unitarily equivalent to a multiplication operator.*

Proof. By Lemma 1.5
$$\mathcal{H} = \oplus_{i\in I}\mathcal{H}_i$$
where I is some index set and H_i is an invariant subspace of A which contains a cyclic vector. Denote by A_i the restriction of the operator A to $\mathcal{H}_i$. By Theorem 1.6 there exists a probability measure μ_i and a unitary
$$U_i : L^2(\sigma(A_i), \mu_i) \to \mathcal{H}_i$$
such that $U_i^{-1}A_iU_i$ is a multiplication operator M_{f_i}. Now we consider the space
$$\mathcal{S} = \oplus_{i\in I}L^2(\sigma(A_i), \mu_i)$$
and define
$$V = \oplus_{i\in I}U_i : \mathcal{S} \to \mathcal{H}\ .$$
Now
$$V^{-1}AV = \oplus_{i\in I}U_i^{-1}A_iU_i = \oplus_{i\in I}M_{f_i}\ .$$
□