1. Spectral theory of bounded self-adjoint operators

In the essential ideas I follow the book of Robert J. Zimmer, "Essential results in functional analysis", a book that I recommend highly.

Definition 1.1. *Let $T : \mathcal{H} \to \mathcal{H}$ by a linear and bounded operator. A number $z \in \mathbb{C}$ is in the* **resolvent set of** *T if the operator $T - zI$ is one-to-one and onto, i.e., is invertible on $\mathcal{H}$. By the open mapping theorem the operator*

$$(T - zI)^{-1} : \mathcal{H} \to \mathcal{H}$$

is bounded and is called the **resolvent of** *T.*

Definition 1.2. *The complement of the resolvent set, $\sigma(T)$, is called the spectrum of T.*

There are two ways that an operator $T - zI$ fails to be invertible. If $T - zI$ is not one-to-one then there exists a vector $f \in \mathcal{H}$ so that

$$(T - zI)f = 0 \ ,$$

which means that f is an eigenvector of T. The set of all eigenvalues is the point spectrum of T and is denoted by $\sigma_p(T)$.

There is, however, a more subtle way in which $T - zI$ fails to be invertible. It could be that $T - zI$ is not onto.

Example Let $\mathcal{H} = L^2(0, 1)$ and $Tf(x) = xf(x)$. Then $T - zI$ is not onto for any $z \in [0, 1]$. Note that

$$(T - zI)^{-1}f(x) = \frac{f(x)}{x - z}$$

and it is natural to think that this operator exists. It is, however, not defined for all $f \in \mathcal{H}$. Simply consider

$$(T - zI)^{-1}1 = \frac{1}{x - z} \notin \mathcal{H}$$

for $z \in [0, 1]$. say that the operator exists in the form

$$(T - zI)^{-1} : Ran(T - zI) \to \mathcal{H}$$

but it is certainly not a bounded operator.

Definition 1.3. *The set of all numbers $z \in \mathbb{C}$ for which $T - zI$ is not onto, but the range of $T - zI$ is dense in $\mathcal{H}$ is called the* **continuous spectrum** *of T.*

Another possibility how $T - zI$ could fail to be invertible is that the range of $T - zI$ is not dense, i.e., there is a non-zero vector $g \in H$ that is perpendicular to the range of $T - zI$. In this case we say that z is in the **residual spectrum of** T.

Example Consider the Hilbert space ℓ^2 consisting of all squaresummable sequences. Consider the linear operator

$$T : \ell^2 \to \ell^2$$

defined by

$$T(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$$

that is, we consider the operator that shifts the sequence $(x - 1, x_2, \dots)$ to the right and replaces the first element by 0. The operator T does not have any eigenvalues. The range of $T = T - 0I$ is not dense in ℓ^2. Hence 0 is in the residual spectrum of T.

We shall use the notation

$$Ran(T)$$

to denote the range of T and

$$Ker(T)$$

to denote the kernel or null space of T. The space of all bounded operator $T : \mathcal{H} \to \mathcal{H}$ we denote by

$$\mathcal{L}(\mathcal{H})$$

Lemma 1.4. *Let $T \in \mathcal{L}(\mathcal{H})$ and assume that for all $f \in \mathcal{H}$*

$$\|Tf\| \geq c\|f\|$$

for some $c > 0$. Then $Ran(T)$ is closed. In particular if $Ran(T)$ is dense in $\mathcal{H}$, then $Ran(T) = \mathcal{H}$ and hence T is invertible.

Proof. Let $g_n \in Ran(T)$ be a sequence that converges to $g \in \mathcal{H}$. We have to show that $g \in Ran(T)$. Since $g_n = Tf_n$ for some f_n, we have that

$$\|g_n - g_m\| = \|T(f_n - f_m)\| \geq c\|f_n - f_m\| \ .$$

Since g_n converges, it is a Cauchy sequence and hence f_n is also a Cauchy sequence and hence convergent to some element $f \in \mathcal{H}$. Since T is bounded it is continuous and hence

$$g = \lim_{n\to\infty} g_n = \lim_{n\to\infty} Tf_n = Tf$$

and $g \in Ran(T)$. If $Ran(T)$ is dense and closed in $\mathcal{H}$, then $Ran(H) = \mathcal{H}$. Hence T is onto. Moreover, since $Tf = 0$ implies that $0 = \|Tf\| \geq c\|f\|$ we find that T is one-to-one. Thus, T is invertible. For $g \in \mathcal{H}$ there exists $f \in \mathcal{H}$ so that $Tf = g$. Hence

$$\|g\| = \|Tf\| \geq c\|f\| = c\|T^{-1}g\| \ .$$

Hence $\|T^{-1}\| \leq \frac{1}{c}$. □

Recall that a bounded operator $T : \mathcal{H} \to \mathcal{H}$ is self-adjoint if $T = T^*$. A bounded operator $U : \mathcal{H} \to \mathcal{H}$ is unitary if it is onto and $\|Uf\| = \|f\|$. This is equivalent to saying that $U^*U = UU^* = I$.

Theorem 1.5. *If T is self-adjoint, then its spectrum $\sigma(T) \subset \mathbb{R}$. If U is unitary, then $\sigma(U) \subset \mathbb{S}^1$, the unit circle.*

Proof. Let $z = x + iy$ with $y \neq 0$. Then, using the fact that $T = T^*$ a simple computation shows that

$$\|Tf - zf\|^2 = \|(T - xI)f - iyf\|^2 = \|(T - xI)f\|^2 + y^2\|f\|^2 \ .$$

Hence,

$$\|Tf - zf\|^2 \geq y^2\|f\|^2$$

In particular, this means that $Ker(T - zI) = \{0\}$. Next we show that $Ran(T - zI)$ is dense. If not, there exists a vector $f \neq 0$ orthogonal to $Ran(T - zI)$. This means for all $g \in \mathcal{H}$

$$0 = \langle (T - zI)g, f\rangle = \langle g, (T - \overline{z}I)f\rangle$$

which implies that $(T - \overline{z}I)f = 0$. Since $Ker(T - \overline{z}I) = \{0\}$, $f = 0$, a contradiction. Thus, by Lemma 1.4 we have that $Ran(T - zI) = \mathcal{H}$ and $(T - zI)^{-1}$ is a bounded operator, in fact

$$\|(T - zI)^{-1}\| \leq \frac{1}{|y|} \ ,$$

and $\sigma(T) \subset \mathbb{R}$.

If U is unitary, pick $z = re^{i\phi}$, with $r \neq 1$. Then

$$\begin{aligned} \|(U - zI)f\|^2 &= \|Uf\|^2 - \langle Uf, zf\rangle - \langle zf, Uf\rangle + r^2\|f\|^2 \\ &= (1 + r^2)\|f\|^2 - 2\mathrm{Re}(\overline{z}\langle Uf, f\rangle) \\ &\geq (1 + r^2)\|f\|^2 - 2r\|f\|^2 = (1 - r)^2\|f\|^2 \ . \end{aligned}$$

Once more this shows that $U - zI$ is one-to-one. Likewise, by the same reasoning it follows that $(U^* - \overline{z}I)$ is one -to one.

If $f \perp Ran(U - zI)$, $f \neq 0$, we find for all $g \in \mathcal{H}$

$$0 = \langle (U - zI)g, f\rangle = \langle g, (U^* - \overline{z}I)f\rangle$$

that is, $(U^* - \overline{z}I)f = 0$. Which is a contradiction. Again, by Lemma 1.4 we find that $(U - zI)^{-1}$ is a bounded operator and

$$\|(U - zI)^{-1}\| \leq \frac{1}{|1 - r|} \ .$$

□

If A is a matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and if $p(x)$ is a polynomial then the eigenvalues of $p(A)$ are given by $p(\lambda_1), \ldots, p(\lambda_n)$. This fact continuous to hold for bounded operators. It works on Banach spaces also; the following proof makes this clear.

Theorem 1.6. *Let $T : \mathcal{H} \to \mathcal{H}$ be a bounded operator and p a polynomial. Then $\sigma(p(T)) = p(\sigma(T))$.*

Proof. If $\lambda \in \sigma(p(T))$ then $p(T) - \lambda I$ is not invertible. The polynomial has the roots $\lambda_1, \ldots, \lambda_n$, i.e., $p(x) - \lambda = a(x - \lambda_1)\cdots(x - \lambda_n)$, where $a \neq 0$ is a constant. Hence

$$p(T) - \lambda I = a(T - \lambda_1 I)\cdots(T - \lambda_n I)$$

and it follows that at least one of the factors is not invertible, say $(T - \lambda_k I)$. Hence $\lambda_k \in \sigma(T)$ and $p(\lambda_k) - \lambda = 0$ and so $\lambda \in p(\sigma(T))$. Now, let $\lambda \in p(\sigma(T))$, i.e., $p(\mu) = \lambda$ for some $\mu \in \sigma(T)$. We have to show that $p(T) - \lambda I$ is not invertible. As before

$$p(T) - \lambda I = a(T - \mu I)(T - \lambda_2 I)\cdots(T - \lambda_n I) = a(T - \lambda_2 I)\cdots(T - \lambda_n I)(T - \mu I)$$

Since $\mu \in \sigma(T)$ then either $(T - \mu I)$ is not onto or $(T - \mu I)$ is not one - to -one or not onto and not one-to-one. If $(T - \mu I)$ is not onto then the first equation above shows that $p(T) - \lambda I$ is not onto. If $(T - \mu I)$ is not one-to-one, then the second equation shows that it $p(T) - \lambda I$ is not one-to-one. Hence $\lambda \in \sigma(p(T))$. □

In the following outline of spectral theory it is advantageous to explain that part of the theory does not depend on the underlying Hilbert space. This part depends only on very general properties of bounded operators and all that is used is the notion of a Banach Algebra.

Definition 1.7. *A Banach algerbra $\mathcal{B}$ is a Banach space, i.e., a complete normed space with the following properties. For any two elements x, y there exists the product xy which is again an element of $\mathcal{B}$. Moreover we have that*

$$(\alpha x + \beta y)z = \alpha xz + \beta yz \ , \alpha, \beta \in \mathbb{C}, x, y, z \in \mathcal{B} \ ,$$

and

$$z(\alpha x + \beta y) = \alpha zx + \beta zy \ , \alpha, \beta \in \mathbb{C}, x, y, z \in \mathcal{B} \ .$$

We further require that

$$\|xy\| \le \|x\|\|y\| \ .$$

If $\mathcal{B}$ has an identity I then we also require that $\|I\| = 1$.

Obviously the space of bounded linear operators on a Hilbert space form a Banach algebra with identity. Note that we do not assume the Banach algebra to be commutative.

Definition 1.8. *For $\mathcal{B}$ a Banach algebra with identity we define the spectrum of $x \in \mathcal{B}$ by*

$$\sigma(x) = \{z \in \mathbb{C} : (x - zI) \text{ is not invertible}\} \ .$$

As before we call the compelement of $\sigma(x)$ the resolvent set of x, $\rho(x)$.

A few simple things about $\rho(x)$ and $\sigma(x)$ can be said right away.

Theorem 1.9. *Let $\mathcal{B}$ be a Banach algebra with identity and $x \in \mathcal{B}$. Then:*
a) The set $\{z \in \mathbb{C} : |z| > \|x\|\}$ is a subset of $\rho(x)$, in particular $\rho(x)$ is not empty.
b) $\rho(x)$ is open and hence $\sigma(x)$ is closed.

Proof. Consider $z \in \mathbb{C}$ with $|z| > \|x\|$. Then

$$x - zI = -z(I - \frac{x}{z})$$

and since $\|\frac{x}{z}\| < 1$ the Neumann series

$$-\frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{x}{z}\right)^k \tag{1}$$

converges in the norm of $\mathcal{B}$ and furnishes the inverse of $x - zI$. If $z_0 \in \rho(x)$ then write

$$x - zI = (x - z_0 I) - (z - z_0)I = (x - z_0 I)\left[I - (z - z_0)(x - z_0 I)^{-1}\right] \ .$$

Hence for

$$|z - z_0| < \|(x - z_0 I)^{-1}\|^{-1}$$

the operator $x - zI$ is invertible by the Neumann series. □

Theorem 1.10 (Spectral radius)**.** *Set*

$$r(x) = \sup\{|z| : z \in \sigma(x)\} \ .$$

Then

$$r(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}}$$

and the limit exists.

The existence of the limit will follow from the lemma.

Lemma 1.11. *Let $f(n)$ be a subadditive function, i.e., for all n, m positive integers*

$$f(n+m) \le f(n) + f(m) \ .$$

Then the limit

$$L = \lim_{n\to\infty} \frac{f(n)}{n}$$

exists in the sense that it might equals $-\infty$. In any case it is given by

$$L = \inf_n \frac{f(n)}{n} \ .$$

Proof. We set

$$L = \inf_n \frac{f(n)}{n} \ .$$

Pick $\varepsilon > 0$ arbitrary. We have to show that

$$\limsup_{n\to\infty} \frac{f(n)}{n} \le L + \varepsilon \ .$$

Pick a positive integer p so that

$$\frac{f(p)}{p} < L + \varepsilon$$

and let n_j be any sequence of positive integers tending to ∞. We divide

$$n_j = m_j p + r_j$$

where the remainder $r_j < p$. we can now estimate

$$\frac{f(n_j)}{n_j} = \frac{f(m_j p + r_j)}{m_j p + r_j} \le \frac{m_j}{m_j p + r_j} f(p) + \frac{1}{m_j p + r_j} f(r_j)$$

Since r_j can take at most p values, $f(r_j)$ is uniformly bounded. As $n_j \to \infty$, m_j must tend to infinity too and hence we have that

$$\limsup_{j\to\infty} \frac{f(n_j)}{n_j} \le \frac{f(p)}{p} < L + \varepsilon$$

and the lemma is proved. □

Proof. Proof of Theorem 1.10: For $n = 1, 2, \ldots$ set

$$f(n) = \log \|x^n\|$$

It is easy to see, using that $\|x^n x^m\| \le \|x^n\| \|x^m\|$, that $f(n)$ satisfies

$$f(n+m) \le f(n) + f(m) \ .$$

Hence, from the lemma above we learn that

$$R(x) := \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}}$$

exists. Returning with this information to (1) we find that the series

$$\frac{1}{|z|} \sum_{k=0}^{\infty} \frac{\|x^k\|}{|z|^k}$$

converges for all $|z|$ with

$$\frac{R(x)}{|z|} < 1$$

and diverges for all z with

$$\frac{R(x)}{|z|} > 1 \ ,$$

by the root test. If $z \in \sigma(x)$ we have that necessarily the series (1) diverges and hence for all $z \in \sigma(x)$, $R(x) > |z|$ and therefore $R(x) \ge r(x)$. If, on the other hand $|z| > r(x)$ then $z \in \rho(x)$. Consider the function

$$f(z) = z(I - zx)^{-1} \ .$$

This function is analytic for z with $|z| < \frac{1}{r(x)}$. We shall use the analogue of a well known theorem in complex variables. If a function f is holomorphic in the unit disk then its power series with respect to the origin has radius of convergence equal to 1. The same holds for holomorphic functions with values in a Banach algebra. The proof uses Cauchy's integral formula and I'll try to publish separate notes on this point where I trace the steps in this proof for holomorphic functions with values in a Banach algebra.

Now the function under consideration has the series expansion

$$z(I - zx)^{-1} = z\sum_{k=0}^{\infty} z^k x^k$$

and, according to what has been said, it converges uniformly, i.e.,

$$\sum_{k=0}^{\infty} |z|^k \|x^k\|$$

converges for all $|z| < \frac{1}{r(x)}$. Thus, The radius of convergence $R(x) \le r(x)$ and hence $R(x) = r(x)$. □

A simple corollary is

Corollary 1.12. *If $\mathcal{H}$ is a Hilbert space and $A \in \mathcal{L}(\mathcal{H})$ is a self adjoint operator then*

$$\|A\| = r(A) \ .$$

Proof. Recall that

$$\|A\|^2 = \sup_{\|f\|=1} \|Af\|^2 = \sup_{\|f\|=1} \langle f, A^*Af\rangle = \|A^*A\|$$

where the last inequality follows from Theorem 3.2 in the section on bounded operators. Since $A = A^*$ we have that $\|A^2\| = \|A\|^2$ and hence

$$\|A^{2^n}\| = \|A\|^{2^n} \ .$$

Hence

$$\|A\| = \|A^{2^n}\|^{\frac{1}{2^n}} \to r(A)$$

as $n \to \infty$.

□

One can easily find examples of operators for which the spectral radius is 0. This operators play in some sense the role of nil-potent matrices. So far we do not know whether the spectrum is empty or not. Oddly, this seems to be a bit more difficult.

Theorem 1.13. *Let $x \in \mathcal{B}$, where $\mathcal{B}$ is a Banach algebra. The the spectrum of x, $\sigma(x)$ is not empty and compact.*

Proof. Suppose it is empty, then $\rho(x) = \mathbb{C}$, i.e., $(x - zI)^{-1}$ exists for all $z \in \mathbb{C}$. If $z_0 \in \mathbb{C}$ then we have that for all

$$|z - z_0| < \|(x - z_0I)^{-1}\|^{-1}$$

that

$$(x - zI)^{-1} = (x - z_0I)^{-1}\sum_{k=0}^{\infty}(z - z_0)^k(x - z_0I)^{-k} \ ,$$

where the series converges in norm. Thus we see that $(x - zI)^{-1}$ is an analytic function with values in $\mathcal{B}$. Next we show that

$$\|(x - zI)^{-1}\|$$

is a subharmonic function, i.e.,

$$\frac{1}{2\pi}\int_0^{2\pi} \|(x - zI - re^{i\phi}I)^{-1}\|d\phi \geq \|(x - zI)^{-1}\|$$

for r sufficiently small. In fact if $r\|(x - zI)^{-1}\| < 1$ we have that

$$(x - ze - re^{i\phi}I)^{-1} = (x - zI)^{-1}\sum_{k=0}^{\infty}(re^{i\phi})^k(x - zI)^{-k}$$

so that

$$\int_0^{2\pi}(x - zI - re^{i\phi}I)^{-1}d\phi = (x - zI)^{-1}\sum_{k=0}^{\infty}\int_0^{2\pi}(re^{i\phi})^k d\phi(x - zI)^{-k} = 2\pi(x - zI)^{-1}$$

where the interchange of the integral and the sum does not pose a problem. Now

$$\|(x - zI)^{-1}\| = \frac{1}{2\pi}\|\int_0^{2\pi}(x - zI - re^{i\phi}I)^{-1}d\phi\| \leq \frac{1}{2\pi}\int_0^{2\pi}\|(x - zI - re^{i\phi}I)^{-1}\|d\phi$$

which is what we wanted to show. It follows from (1) that $\|(x - zI)^{-1}\|$ is uniformly bounded outside the closed disk of radius $\|x\|$, in fact as $|z| \to \infty$ $\|(x - zI)^{-1}\| \to 0$. Since $\|x - zI)^{-1}\|$ is continuous it is uniformly bounded in the closed disk of radius $\|x\|$ and hence it attains its maximum at some point z_0. Pick any other point z_1 and consider the segment

$$z(t) = (1 - t)z_0 + tz_1$$

For $0 \leq t < \frac{1}{\|(x - z_0I)^{-1}\|}$ the function $\|(x - z(t)I)^{-1}\|$ is constant. This follows from

$$\|(x - z_0I)^{-1}\| \leq \frac{1}{2\pi}\int_0^{2\pi}\|(x - z_0I - re^{i\phi}I)^{-1}\|d\phi \leq \frac{1}{2\pi}\int_0^{2\pi}\|(x - z_0I)^{-1}\|d\phi = \|(x - z_0I)^{-1}\|$$

for all r with $r\|(x - z_0I)^{-1}\| < 1$. Hence for t with $\|(x - z_0I)^{-1}\| < 1$ the function $\|(x - z(t)I)^{-1}\|$ is constant. Since $\|(x - z(t)I)^{-1}\|$ is continuous we know that the set

$$C = \{0 \leq t \leq 1 : \|(x - z(t)I)^{-1}\| = \|(x - z_0I)^{-1}\|\}$$

is a closed set. It is also open since if $t_0 \in C$ by repeating the same argument we find that for δ with $\delta\|(x - z(t_0)I)^{-1}\| < 1$, $(t_0 - \delta, t_0 + \delta) \cap [0, 1] \subset C$. Since the interval $[0, 1]$ is connected we must have that $C = \emptyset$ or $C = [0, 1]$. The former is not possible, since $z_0 \in C$. Hence $C = [0, 1]$. Thus, we conlude that the function

$$\|(x - zI)^{-1}\|$$

is a constant. Since $\|(x - zI)^{-1}\| \to 0$ as $|z| \to \infty$, this is a contradiction and $\sigma(x)$ is not empty. $\square$