1. Prep-Final A

Problem 1: Find the speed, the tangential acceleration and the normal acceleration for the motion
$$\vec{r}(t) = (t, t^2, t^2) \ .$$
Compute also the curvature of the corresponding curve as a function of t.

The velocity, resp. accelaration is
$$\vec{v}(t) = (1, 2t, 2t) \ , \ \vec{a} = (0, 2, 2) \ ,$$
and
$$|\vec{v}|$$
is the speed. The tangential acceleration is
$$a_T = \frac{d}{dt}|\vec{v}(t)| = \frac{d}{dt}\sqrt{1+8t^2} = \frac{8t}{\sqrt{1+8t^2}}$$
The normal acceleration is
$$a_N = \sqrt{|\vec{a}|^2 - a_T^2} = \sqrt{8 - \frac{64t^2}{1+8t^2}} = \frac{\sqrt{8}}{\sqrt{1+8t^2}}$$
The curvature can be found using the formula
$$\kappa(t) = \frac{|\vec{a} \times \vec{v}|}{|\vec{v}|^3} = \frac{\sqrt{8}}{(1+8t^2)^{3/2}} \ .$$
Here are some explanations: The tangential and normal acceleration are defined by
$$\vec{a} = a_T\vec{T} + a_N\vec{N} \ ,$$
where
$$\vec{T} = \frac{\vec{v}}{|\vec{v}|} \ , \vec{N} = \frac{dT}{ds} = \frac{dT}{dt}\frac{dt}{ds} = \frac{\frac{dT}{dt}}{|\vec{v}|}$$
and s is the length parametrization. Now compute:
$$\vec{a} = \frac{d}{dt}\vec{v} = \frac{d}{dt}(\vec{T}\frac{ds}{dt}) = \frac{d^2s}{dt^2}\vec{T} + \frac{d\vec{T}}{ds}(\frac{ds}{dt})^2$$
so that
$$a_T = \frac{d^2s}{dt^2} = \frac{d|\vec{v}|}{dt} \ ,$$
and
$$a_N = \kappa(\frac{ds}{dt})^2 \ ,$$
recalling that the curvature is given by
$$\kappa = |\frac{d\vec{T}}{ds}| \ .$$
From this it follows that
$$\kappa = \frac{|\vec{a} \times \vec{v}|}{|\vec{v}|^3} \ .$$

The formula

$$\vec{a} = \frac{d^2s}{dt^2}\vec{T} + \kappa\left(\frac{ds}{dt}\right)^2\vec{N}$$

is useful and intuitive.

Problem 2: Find the moment of inertia with respect to the x axis of a thin shell of mass δ that is in the first quadrant of the xy plane and bounded by the curve $r^2 = \sin 2\theta$.

The moment of inertia with respect to the x axis is

$$\delta \int_{Region} y^2 dxdy \ .$$

It is reasonable to work this integral in polar coordinates. Note that $\sin 2\theta > 0$ only if $0 \le \theta \le \pi/2$ and $\pi \le \theta \le 3\pi/2$. Being in the first quadrant requires $0 \le \theta \le \pi/2$. The moment of inertia with respect to the x axis is now the integral

$$\delta \int_0^{\pi/2} \int_0^{\sqrt{\sin 2\theta}} (r\sin\theta)^2 r dr d\theta \ .$$

The distance of the point (x, y) to the x axis is y^2. Integrating with respect to r yields

$$\frac{\delta}{4}\int_0^{\pi/2} (\sin 2\theta)^2 (\sin\theta)^2 d\theta = \delta \int_0^{\pi/2} (\sin\theta)^4 (\cos\theta)^2 d\theta \ .$$

$$= \delta \int_0^{\pi/2} (\sin\theta)^4 d\theta - \delta \int_0^{\pi/2} (\sin\theta)^6 d\theta = \delta\frac{3\pi}{2^4} - \delta\frac{5\pi}{2^5} = \delta\frac{\pi}{2^5} \ .$$

Problem 3: Compute the center of mass of a thin shell that is formed by the cone $(z-2)^2 = x^2 + y^2$, $0 \le z \le 2$.

The following solves the wrong problem, namely for the solid cone.

The tip of the cone is at $z = 2$ and the base is a disk of radius 2. We use cylindrical coordinates. By symmetry $x_{CM} = y_{CM} = 0$. Now

$$z_{CM} = \frac{\int_0^{2\pi}\int_0^2\int_0^{2-r} z dz r dr d\theta}{\int_0^{2\pi}\int_0^2\int_0^{2-r} dz r dr d\theta} \ .$$

The numerator is

$$\frac{1}{2}\int_0^{2\pi}\int_0^2 (2-r)^2 r dr d\theta = \pi \int_0^2 (2-r)^2 r dr = \frac{4\pi}{3}$$

and the denominator is

$$\int_0^{2\pi}\int_0^2 (2-r) r dr d\theta = 2\pi \int_0^2 (2-r) r dr = \frac{8\pi}{3}$$

so that

$$z_{CM} = \frac{1}{2} \ .$$

Now the solution for the problem as posed:

Again, we have that $x_{CM} = y_{CM} = 0$, as before. The z coordinate is

$$z_{CM} = \frac{\int_{\text{Surface}} z d\sigma}{\int_{\text{Surface}} d\sigma}$$

the density δ cancels. We have to parametrize the cone, and we use conveniently cylindrical coordinates,

$$\vec{r}(\theta, r) = (r\cos\theta, r\sin\theta, 2-r)$$

noting that on the cone $z = 2 - r$. The tangent vectors are given by

$$\vec{r}_r = (\cos\theta, \sin\theta, -1)$$

and

$$\vec{r}_\theta = (-r\sin\theta, r\cos\theta, 0)\ .$$

The surface element is

$$d\sigma = |\vec{r}_r \times \vec{r}_\theta| dr d\theta = |(r\cos\theta, r\sin\theta, r)| dr d\theta = \sqrt{2} r dr d\theta\ .$$

Now we integrate:

$$\int_{\text{Surface}} d\sigma = \int_0^{2\pi} \int_0^2 \sqrt{2} r dr d\theta = 4\pi\sqrt{2}\ .$$

$$\int_{\text{Surface}} z d\sigma = \int_0^{2\pi} \int_0^2 (2-r)\sqrt{2} r dr d\theta = 2\pi\sqrt{2} \int_0^2 (2-r) r dr = \frac{8\pi}{3}\sqrt{2}\ .$$

Hence,

$$z_{CM} = \frac{2}{3}$$

Note that the Center of Mass is higher for the shell than for the solid, which is reasonable.

Problem 4: Compute the line integral of the vector field

$$\vec{F} = (xyz + 1, x^2 z, x^2 y) e^{xyz}$$

along the curve given in parametrized form by

$$\vec{r}(t) = (\cos t, \sin t, t)\ ,\ 0 \le t \le \pi\ .$$

The line integral looks complicated and it is advisable to use Stokes's theorem. Computing the curl of $\vec{F}$ yields $(0,0,0)$ and hence, by Stokes's theorem the line integral depends only on the end points. The straight line that connects these two points is

$$\vec{r}(t) = (1-t)(1,0,0) + t(-1,0,\pi) = (1-2t, 0, t\pi)\ ,\ 0 \le t \le 1\ .$$

We compute

$$\vec{F} \cdot \vec{r}' = (1, (1-2t)^2 t\pi, 0) \cdot (-2, 0, \pi) = -2$$

and integrating this from 0 to 1 yields

$$-2\ .$$

With a little bit of guesswork one finds that

$$\vec{F} = \nabla f\ ,\ f = x e^{xyz}$$

and

$$f(-1,0,\pi) - f(1,0,0) = -1 - 1 = -2 \ .$$

Problem 5: Use the divergence theorem to compute the outward flux of the vector field

$$\vec{F} = (x^2, y^2, z^2)$$

through the cylindrical can that is bounded on the side by the cylinder $x^2 + y^2 = 4$, bounded above by $z = 1$ and below by $z = 0$.

Again, we invoke an integral theorem, but this time the divergence theorem. One computes easily

$$\text{div}\vec{F} = 2(x + y + z)$$

and we have to integrate this over the cylinder. Using cylindrical coordinates

$$2\int_0^{2\pi}\int_0^2\int_0^1 [r(\cos\theta + \sin\theta) + z]dzrdrd\theta = 4\pi \ .$$

One can try to compute the flux directly. For the flux through the top one has to integrate

$$(x^2, y^2, 1) \cdot (0, 0, 1)$$

over the disk of radius 2, which yields 4π. The bottom disk is particularly easy since the normal vector is $(0, 0, -1)$ and the vector field is $(x^2, y^2, 0)$ so that the dot product vanishes. Hence there is no contribution. It remains to compute the flux through the side. The parametrization of the cylinder is

$$\vec{r}(\theta, z) = (2\cos\theta, 2\sin\theta, z)$$

so that

$$\vec{r}_\theta = (-2\sin\theta, 2\cos\theta, 0) \ , \ \vec{r}_z = (0, 0, 1)$$

and

$$\vec{r}_\theta \times \vec{r}_z = 2(\cos\theta, \sin\theta, 0)$$

which obviously points outward. Now

$$\vec{F} \cdot \vec{n}d\sigma = ((2\cos\theta)^2, (2\sin\theta)^2, z^2) \cdot 2(\cos\theta, \sin\theta, 0)d\theta dz = 8((\cos\theta)^3 + (\sin\theta)^3)d\theta dz$$

and

$$8\int_0^1\int_0^{2\pi}((\cos\theta)^3 + (\sin\theta)^3)d\theta dz = 0 \ .$$