1. Prep-Final B

Problem 1: Find the parametric equations of the line that is tangent to the curve

$$\vec{r}(t) = (e^t, \sin t, \ln(1-t))$$

at $t = 0$.

One point on the line is

$$(1, 0, 0) \ .$$

The tangent at this point is

$$\vec{r}'(0) = (1, 1, -1)$$

so that the line tangent is given by

$$(1, 0, 0) + s(1, 1, -1) \ , s \in \mathbb{R} \ .$$

Problem 2: Find the minimum cost area of a rectangular solid with volume 64 cubic inches, given that the top and sides cost 4 cents per square inch and the bottom costs 7 cents per square inch. Just set up the equations using Lagrange multipliers, you do not have to solve them.

The sides have length a, b, c. The top side is ab the bottom is also ab and the sides are ac, bc. Hence the cost of these is in total

$$11ab + 4 \cdot 2(ac + bc) \ .$$

The volume is

$$abc = 64 \ .$$

Now we use Lagrange multipliers. Here $f(a, b, c) = 11ab + 8(ac + bc)$ and $g(a, b, c) = abc - 64$.

$$\nabla f = (11b + 8c, 11a + 8c, 8(a + b)) = \lambda \nabla g = \lambda(bc, ac, ab) \ ,$$

or

$$\begin{aligned} 11b + 8c &= \lambda bc \\ 11a + 8c &= \lambda ac \\ 8(a + b) &= \lambda ab \end{aligned}$$

which together with $abc = 64$ forms 4 equations with 4 unknowns. Although not required they can be solved. Multiplying the the first by a, the second by b etc. we find

$$(11b + 8c)a = (11a + 8c)b = 8c(a + b) \ .$$

The first equality sign yields $ac = cb$, the second $11ab = 8ac$. None of the numbers can be zero since $abc = 64$. Hence we have that

$$a = b \ , \ 11b = 8c$$

so that

$$b = a \ , c = \frac{11}{8} a$$

and

$$a^3 \frac{11}{8} = 64$$

$$a = \frac{8}{(11)^{1/3}} \ .$$

Problem 3: Compute the average of the function x^4 over the sphere centered at the origin whose radius is $R > 0$.

The average is given by the formula

$$\frac{\int_{S_R} x^4 d\sigma}{\int_{S_R} d\sigma} \ .$$

The denominator is the surface area of the sphere of radius R and hence $4\pi R^2$. For the other integral we resort to spherical coordinates

$$x = R \sin\phi \cos\theta \ , \ y = R \sin\phi \sin\theta \ , \ z = R \cos\phi \ ,$$

and $0 \le \phi \le \pi$, and $0 \le \theta < 2\pi$. The surface area element is

$$d\sigma = R^2 \sin\phi d\phi d\theta \ .$$

Then we find

$$\int_{S_R} |x|^4 d\sigma = \int_0^{2\pi} \int_0^{\pi} R^4 \sin\phi^4 \cos\theta^4 R^2 \sin\phi d\phi d\theta$$

$$= R^6 \int_0^{2\pi} \cos\theta^4 d\theta \int_0^{\pi} \sin\phi^5 d\phi \ .$$

Now,

$$\int_0^{\pi} \sin\phi^5 d\phi = \frac{4}{5} \int_0^{\pi} \sin\phi^2 d\phi = \frac{4}{5}\frac{2}{3} \int_0^{\pi} \sin\phi d\phi = \frac{16\pi}{15} \ .$$

Likewise

$$\int_0^{2\pi} \cos\theta^4 d\theta = \frac{3\pi}{4}$$

so that we get for the average

$$\frac{R^4}{5} \ .$$

A cleverer way would have been to not that the average over x^4 is the same as the one for z^4. The integral for this is

$$R^6 \int_0^{2\pi} \int_0^{\pi} \cos^4\phi \sin\phi d\phi d\theta = R^6 2\pi \int_{-1}^{1} u^4 du = \frac{R^6 4\pi}{5}$$

which leads to the result.

Problem 4: Compute the flux

$$\int_S \vec{F} \cdot \vec{n} d\sigma$$

where S is the hemisphere $x^2 + y^2 + z^2 = 4, z \geq 0$, $\vec{n}$ points toward the origin and

$$\vec{F} = (x(z-y), y(x-z), z(y-x)) \ .$$

Despite the fact that the surface is not a closed one one can still try to use the divergence theorem. We close it by adding the disk at $z = 0$ that closes the hemi-sphere. Let's call this closed surface T which is the union of the surface S and the bottom B. Now

$$\int_T \vec{F} \cdot \vec{n} d\sigma = \int_V \text{div} \vec{F} dv$$

where V is the interior of the surface T. Note that the normal we use in the theorem is the outward normal! The divergence is

$$\text{div} \vec{F} = z - y + x - z + y - x = 0 \ .$$

Hence

$$\int_S \vec{F} \cdot \vec{n} d\sigma = - \int_B \vec{F} \cdot \vec{n} d\sigma$$

where, I repeat, the normal vectors are the outward normal ones. But our integral is the with the inward normal which is the *negative* of the one with the outward normal. Hence it remains to compute

$$\int_B \vec{F} \cdot \vec{n} d\sigma \ .$$

The vector field at the bottom is given by

$$\vec{F} = (-xy, xy, 0)$$

the normal outward vector is

$$(0, 0, -1) \ .$$

Thus the dot product vanishes and hence we have that

$$\int_S \vec{F} \cdot \vec{n} d\sigma = 0 \ .$$

Problem 5: Compute the line integral $\int_C \vec{F} \cdot \vec{dr}$ where C is the curve given by the intersection of the sphere $x^2 + y^2 + z^2 = 4$ and the plane $z = -y$, counterclockwise when viewed from above, and

$$\vec{F} = (x^2 + y, x + y, 4y^2 - z) \ .$$

We start by writing Stokes's theorem

$$\int_C \vec{F} \cdot \vec{dr} = \int \int_S \text{curl} \vec{F} \cdot \vec{n} d\sigma \ .$$

Here

$$\text{curl} \vec{F} = (8y, 0, 0) \ .$$

The next problem is how to choose the surface whose boundary is the curve C. The simplest what comes to mind is the surface formed by the intersection of the sphere with the plane. This is a circle in the plane $z + y = 0$. The normal vector is

$$\vec{n} = \frac{(0,1,1)}{\sqrt{2}}$$

which has together with the curve the right orientation. Now we see that the dot product of curlF with $\vec{n}$ vanishes and hence

$$\int_C \vec{F} \cdot \vec{dr} = 0$$