1. Prepquiz 1B Solutions

Problem 1: Consider the curve in the plane given by the motion

$$\vec{r}(t) = \langle t, \frac{2}{3}t^{3/2}\rangle \ , 0 \le t \le 1 \ .$$

a) Compute the length parametrization $s(t)$, i.e., the length of the piece of the curve between $\vec{r}(0)$ and $\vec{r}(t)$.

$$\vec{r}'(t) = \langle 1, t^{1/2}\rangle$$

and $|\vec{r}'(t)| = \sqrt{1+t}$ so that

$$s(t) = \int_0^t \sqrt{1+t}dt = \frac{2}{3}(1+t)^{3/2}|_0^t = \frac{2}{3}[(1+t)^{3/2} - 1]$$

b) Compute the length L of the curve.

$$L = s(1) = \frac{2}{3}[2^{3/2} - 1] \ .$$

c) Compute the inverse function $t(s)$ and find an expression for the curve in terms of the length parametrization. Compute the unit tangent vector and the curvature in this parametrization. What happens at $s = 0$?

$$t(s) = [\frac{3s}{2} + 1]^{2/3} - 1$$

and

$$\vec{u}(s) = \langle [\frac{3s}{2} + 1]^{2/3} - 1, \frac{2}{3}[[\frac{3s}{2} + 1]^{2/3} - 1]^{3/2}\rangle \ .$$

The unit tangent vector is $\vec{T}(s) = \vec{u}'(s)$ and given by

$$\vec{T}(s) = \langle [\frac{3s}{2} + 1]^{-1/3}, [[\frac{3s}{2} + 1]^{2/3} - 1]^{1/2}[\frac{3s}{2} + 1]^{-1/3}\rangle = \langle [\frac{3s}{2} + 1]^{-1/3}, [1 - [\frac{3s}{2} + 1]^{-2/3}]^{1/2}\rangle \ .$$

Further

$$\vec{T}'(s) = \langle -\frac{1}{2}[\frac{3s}{2} + 1]^{-4/3}, \frac{1}{2}[1 - [\frac{3s}{2} + 1]^{-2/3}]^{-1/2}[\frac{3s}{2} + 1]^{-5/3}\rangle$$

$$\kappa(s) = \sqrt{\frac{1}{4}[\frac{3s}{2} + 1]^{-8/3} + \frac{1}{4}[1 - [\frac{3s}{2} + 1]^{-2/3}]^{-1}[\frac{3s}{2} + 1]^{-10/3}}$$

$$= \frac{1}{2}\frac{1}{[\frac{3s}{2} + 1]\sqrt{[\frac{3s}{2} + 1]^{2/3} - 1}}$$

Problem 2: Find the unit tangent and curvature of the curve described by the motion

$$\vec{r}(t) = \frac{1}{2}t^2\vec{i} + \frac{1}{3}t^3\vec{j} + t\vec{k} \ .$$

$$\vec{v}(t) = \vec{r}'(t) = t\vec{i} + t^2\vec{j} + \vec{k}$$

so that

$$|\vec{v}| = \sqrt{1+t^2+t^4}\ .$$

Now

$$\vec{T}(t) = \frac{\vec{v}}{|\vec{v}|} = \frac{t\vec{i} + t^2\vec{j} + \vec{k}}{\sqrt{1+t^2+t^4}}$$

$$\vec{T}'(t) = \frac{\langle 1+t^2+t^4, 2t(1+t^2+t^4), 0\rangle}{(1+t^2+t^4)^{3/2}} - \frac{\langle t^2+2t^4, t^3+2t^5, t+2t^3\rangle}{(1+t^2+t^4)^{3/2}} = \frac{\langle 1-t^4, 2t+t^3, -t-2t^3\rangle}{(1+t^2+t^4)^{3/2}}$$

$$\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{v}(t)|} = \frac{\sqrt{1+5t^2+6t^4+5t^6+t^8}}{(1+t^2+t^4)^2} = \frac{\sqrt{1+4t^2+t^4}}{(1+t^2+t^4)^{3/2}}$$

Another, and easier, way of computing the curvature would be to use the formula

$$\frac{|\vec{v}\times\vec{a}|}{|\vec{v}|^3}\ .$$

We find

$$\vec{a} = \vec{i} + 2t\vec{j}$$

and

$$\vec{v}\times\vec{a} = -2t\vec{i} + \vec{j} + t^2\vec{k}$$

so that

$$|\vec{v}\times\vec{a}| = \sqrt{1+4t^2+t^4}$$

and once more

$$\kappa(t) = \frac{\sqrt{1+4t^2+t^4}}{(1+t^2+t^4)^{3/2}}\ .$$

Problem 3: Find the normal and tangential component of the acceleration of the motion

$$\vec{r}(t) = 3t^2\vec{i} + 4t^2\vec{j} + 5t\vec{k}\ .$$

Recall that

$$\vec{a} = a_T\vec{T} + a_N\vec{N}$$

where

$$a_T = \frac{d|\vec{v}|}{dt} = \frac{d^2s}{dt^2}\ ,\ a_N = \kappa|\vec{v}|^2 = \kappa(\frac{ds}{dt})^2\ .$$

We find

$$\vec{v} = 6t\vec{i} + 8t\vec{j} + 5\vec{k}$$

and

$$|\vec{v}| = \sqrt{25+100t^2} = 5\sqrt{1+4t^2}\ .$$

From this we compute

$$a_T = \frac{d^2s}{dt^2} = \frac{20t}{\sqrt{1+4t^2}}$$

The acceleration is

$$\vec{a} = 6\vec{i} + 8\vec{j}\ .$$

so that

$$|\vec{a}| = 10$$

and therefore

$$a_N = \sqrt{\vec{a}|^2 - a_T^2} = \sqrt{100 - \frac{400t^2}{1+4t^2}} = \frac{10}{\sqrt{1+4t^2}}$$