1. Prepquiz 2 A

Problem 1: The sphere $x^2 + y^2 + z^2 = 14$ and the plane $x + y - z = 0$ intersect in a circle. Find the line tangent to this circle at the point $(1, 2, 3)$.

The gradient of the function $g(x, y, z) = x^2 + y^2 + z^2$ is $\nabla g(x, y, z) = \langle 2x, 2y, 2z\rangle$, so that

$$\nabla g(1, 2, 3) = \langle 2, 4, 6\rangle \ .$$

Likewise, the gradient of the function $f(x, y, z) = x + y - z$ is $\nabla f(x, y, z) = \langle 1, 1, -1\rangle$ so that

$$\nabla f(1, 2, 3) = \langle 1, 1, -1\rangle \ .$$

The plane tangent to the sphere at the point $(1, 2, 3)$ is perpendicular to $\nabla g(1, 2, 3)$. The tangent line we are loooking for is the intersection of these two planes. Hence the direction of this line is perpendicular to $\nabla g(1, 2, 3)$ and $\nabla f(1, 2, 3)$, i.e., perpendicular to both $\langle 2, 4, 6\rangle$ and $\langle 1, 1, -1\rangle$. it is therfore natural to compute the cross product

$$\langle 2, 4, 6\rangle \times \langle 1, 1, -1\rangle = \langle -10, 8, -2\rangle \ .$$

The lines must pass through the point $(1, 2, 3)$ and therefore the tangent line is given by all points of the form

$$(1, 2, 3) + s\langle -10, 8, -2\rangle \ .$$

Problem 2: Find all the circles of radius 1 that are tangent to the curve $\frac{x^2}{4} + y^2 = 2$ at the point $(-2, 1)$.

Any circle of radius 1 is of the form $(x - a)^2 + (y - b)^2 = 1$. The goal is to find a, b so that the circles is tangent to the ellipse at the point $(-2, 1)$. To be tangent at the point $(-2, 1)$ means that the gradients of the function $f(x, y) = \frac{x^2}{4} + y^2$ and of the function $g(x, y) = (x - a)^2 + (y - b)^2$ evaluated at $(-2, 1)$ are parallel. Since

$$\nabla f(x, y) = \langle \frac{x}{2}, 2y\rangle \ , \ \nabla g(x, y) = \langle 2(x - a), 2(y - b)\rangle$$

so that

$$\nabla f(-2, 1) = \langle -1, 2\rangle \ , \ \nabla g(-2, 1) = \langle -4 - 2a, 2 - 2b\rangle \ .$$

That the two vectors are parallel means that

$$\langle -4 - 2a, 2 - 2b\rangle = \lambda\langle -1, 2\rangle$$

which when written out $4 + 2a = \lambda$ and $2 - 2b = 2\lambda$. This means that

$$a = -2 + \frac{\lambda}{2} \ , \ b = 1 - \lambda \ .$$

We also know that $(-2 - a)^2 + (1 - b)^2 = 1$, since the point $(-2, 1)$ must be on the circle. So it must be that

$$1 = (-2 - a)^2 + (1 - b)^2 = \frac{5\lambda^2}{4} \ .$$

so that $\lambda = \pm\frac{2}{\sqrt{5}}$. Thus, there are two circles which touches the ellipse, one on the 'inside' and one on the 'outside'. They are

$$(x + 2 - \frac{1}{\sqrt{5}})^2 + (y - 1 + \frac{2}{\sqrt{5}})^2 = 1$$

and

$$(x + 2 + \frac{1}{\sqrt{5}})^2 + (y - 1 - \frac{2}{\sqrt{5}})^2 = 1$$

Problem 3: Find the distance between the curve $x^2 - xy + y^2 = 1$ and the line $x + y = 10$. What is the point closest to the line?

The line of distance between the curve $x^2 - xy + y^2 = 1$ and the line $x + y = 10$ must be perpendicular to both, the curve and the line. The vector normal to the line, which is the same as the gradient of the function $x+y$, is $\langle 1, 1\rangle$. Thus we have to find a point (u, v) on the curve $x^2 - xy + y^2 = 1$ where the gradient of the function $f(x, y) = x^2 - xy + y^2$ is parallel to $\langle 1, 1\rangle$. Now

$$\nabla f(x, y) = \langle 2x - y, -x + 2y\rangle$$

and we have to solve

$$2x - y = \lambda \ , \ -x + 2y = \lambda$$

for x and y. This is easily done and we get $(x, y) = \lambda(1, 1)$. Now we have to choose λ so that the point $\lambda(1, 1)$ is on the curve, i.e., $\lambda^2 = 1$ and hence $\lambda = \pm 1$. The curve is an ellipse and one point is closest to the line whereas the other is farthest from the line. Now we compute the distance between the line $x + y = 1$ and the point $(1, 1)$. The line passing through the point $(1, 1)$ and is normal to the line $x + y = 10$ is given by

$$(1, 1) + s\langle 1, 1\rangle$$

(Why?) and it intersects the line $x + y = 10$ at the point $(5, 5)$, i.e., $s = 4$. But the distance between the points $(1, 1)$ and $(5, 5)$ is $4\sqrt{2}$. Similarly the distance of the point $(-1, -1)$ to the line is $6\sqrt{2}$ which is larger. Hence the distance is $4\sqrt{2}$ and the point on the ellipse that is closest to the line is $(1, 1)$.