1. Prepquiz 2 B

Problem 1: Consider the function $f(x, y, z) = xy + xz + zy$. At the point $(1, 3, 1)$ find all the direction vectors $\vec{u}$ so that a) $D_{\vec{u}}f(1, 3, 1)$ is as large as possible, b) as small as possible and c) so that $D_{\vec{u}}f(1, 3, 1) = 0$.

We have that

$$\nabla f(x, y, z) = \langle y + z, x + z, x + y \rangle$$

and

$$\nabla f(1, 3, 1) = \langle 4, 2, 4 \rangle \ .$$

The derivative in the direction $\vec{u}$ is $\nabla f(1, 3, 1) \cdot \vec{u}$ and this is largest when the unit vector $\vec{u}$ is parallel to $\nabla f(1, 3, 1)$ in which case $\nabla f(1, 3, 1) \cdot \vec{u} = |\nabla f(1, 3, 1)| = 6$. The vector $\vec{u}$ is given by

$$\vec{u} = \frac{\langle 2, 1, 2 \rangle}{3} \ .$$

The smallest possible value we get when

$$\vec{u} = -\frac{\langle 2, 1, 2 \rangle}{3}$$

in which case $\nabla f(1, 3, 1) \cdot \vec{u} = -6$. Finally, all vectors for which $\nabla f(1, 3, 1) \cdot \vec{u} = 0$ are vectors that are perpendicular to $\langle 4, 2, 4 \rangle$, i.e., all vectors $\langle u, v, w \rangle$ that satisfy

$$4u + 2v + 4w = 0 \ .$$

Problem 2: a) Find the plane tangent to the graph of $f(x, y) = x^4 - 6x^2y^2 + y^4$ at the point $(1, 1, -4)$. b) Find the line normal to the graph of f at the same point.

The graph of $f(x, y)$ can thought of as the level surface

$$f(x, y) - z = 0 \ .$$

The gradient of the function is $\langle f_x, f_y, -1 \rangle$ which is given by

$$\langle 4x^3 - 12xy^2, -12x^2y + 4y^3, -1 \rangle$$

which at the point $(1, 1, -4)$ is

$$\langle -8, -8, -1 \rangle$$

Hence the plane is given by the equation

$$8(x - 1) + 8(y - 1) + (z + 4) = 0 \ .$$

The line normal to the surface that the point $(1, 1, -4)$ is

$$(1, 1, -4) + s\langle 8, 8, 1 \rangle \ .$$

Problem 3: Find the absolute maximum and minimum of the function $f(x, y) = 2x^2 - 4x + y^2 - 4y$ on the closed triangle bounded by the lines $x = 0$, $y = 2$ and $y = 2x$ in the first quadrant. Find all the points in this triangle where these values are attained.

The critical points are given by those points (x, y) where $\nabla f(x, y) = \langle 0, 0\rangle$. Thus we have to solve the equation

$$4x - 4 = 0 \ , 2y - 4 = 0$$

and so $x = 1$ and $y = 2$. The critical point is not in the interior and hence the maximum and the minimum is not attained in the interior of the triangle. Now we have to check the boundary.

1) $x = 0, 0 \le y \le 2$. The function $g(y) := f(0, y) = y^2 - 4y$. Its derivative vanishes also at $y = 2$ which is at the end point of the interval $[0, 2]$. Hence we have to deal with the end points only.

$$f(0, 0) = 0 \ , \ f(0, 2) = -4 \ .$$

2) $y = 2, 0 \le x \le 1$. The function $g(x) := f(x, 2) = 2x^2 - 4x - 4$. Its derivative vanishes at $x = 1$ which is again the endpoint of the interval $[0, 1]$ and hence we only have to consider

$$f(1, 2) = -6 \ , f(0, 2) = -4 \ .$$

3) $y = 2x, 0 \le x \le 1$. The function $g(x) := f(x, 2x) = 6x^2 - 12x$. Its derivative vanishes at $x = 1$ and once more this is an endpoint. The values of $f(x, y)$ at the endpoints we already know (it is a triangle!).

Hence the maximal value of the function on the triangle is attained at $(0, 0)$ and value is 0. Its minimal value of the function on the triangle is -6 attained at the point $(1, 2)$.

Here is a really slick way to solve problem. This is due to the special nature of the function. We may complete the square and get that

$$f(x, y) = 2(x - 1)^2 + (y - 2)^2 - 6 \ .$$

Obviously the function is smallest when $(x, y) = (1, 2)$ which is on the triangle! Now to make the function largest we can argue that $(x - 1)^2 \le 1$ since x can only range between 0 and 1. Further $(y - 2)^2 \le 2$ because y can only range between 0 and 2. Hence

$$f(x, y) \le f(0, 0) = 2 + 4 - 6 = 0 \ .$$