1. Prepquiz 3 A

Problem 1: Find the volume of the solid that is bounded below by the xy plane, on the sides by the planes $y = 0, x = 0$ and $x + y = 1$ and on top by the surface $z = x^2 + y^2$.

This volume is given as

$$\int_R (x^2 + y^2) dx dy$$

where R is the triangle bounded by the lines $x = 0, y = 0$ and $x + y = 1$. Now we work this by first fixing x and integrating with respect to y from 0 to $1 - x$ and then over x from 0 to 1. In terms of a formula

$$\int_0^1 \left(\int_0^{1-x} (x^2 + y^2) dy \right) dx$$

$$= \int_0^1 \left(x^2(1 - x) + \frac{(1 - x)^3}{3} \right) dx = \int_0^1 \left(x^2 - x^3 + \frac{1}{3} - x + x^2 - \frac{1}{3}x^3 \right) dx$$

$$= \int_0^1 \left(2x^2 - \frac{4}{3}x^3 + \frac{1}{3} - x \right) dx = \frac{1}{6}$$

Problem 2: The lemniscate is the curve given by the equation

$$(x^2 + y^2)^2 = 4xy \ .$$

a) Sketch this curve.

Using polar coordinates the curve is given by

$$r^4 = 4r^2 \cos\theta \sin\theta = 2r^2 \sin(2\theta)$$

or

$$r^2 = 2 \sin(2\theta) \ .$$

Since $\sin(2\theta)$ has to be positive $0 \le \theta \le \pi/2$ and $\pi \le \theta \le 3\pi/2$. Thus the curve looks like a figure eight with one leave in the first quadrant and the second leave in the third quadrant.

b) Find the area of the region enclosed by this curve.

The area is given in terms of an integral in polar coordinates. The range of r is given by $0 \le r \le \sqrt{2\sin(2\theta)}$ and the range of θ is given as above, i.e., $0 \le \theta \le \pi/2$ and $\pi \le \theta \le 3\pi/2$. Hence we find for the area A

$$A = \int_0^{\pi/2} \left(\int_0^{\sqrt{2\sin(2\theta)}} r dr \right) d\theta + \int_\pi^{3\pi/2} \left(\int_0^{\sqrt{2\sin(2\theta)}} r dr \right) d\theta = 2 \int_0^{\pi/2} \left(\int_0^{\sqrt{2\sin(2\theta)}} r dr \right) d\theta$$

$$= \int_0^{\pi/2} 2\sin(2\theta) d\theta = \int_0^{\pi} \sin(\theta) d\theta = 2 \ .$$

Problem 3: Find the extremal values of $xy + yz + zx$ given the constraint $x + y + z = 1$. Are these values maxima or minima?

First we set up the Lagrange multiplier scheme: Set $f(x, y, z) = xy + yz + zx$ and $g(x, y, z) = x + y + z - 1$. Then

$$\nabla f = \lambda \nabla g \text{ and } g = 0$$

are the equations which we have to solve. Computing the gradients explicitly

$$y + z = \lambda \ , \ z + x = \lambda \ , \ x + y = \lambda \ , \ x + y + z - 1 = 0$$

Remember that the important variables are x, y, z. We find

$$y + z = \lambda = z + x$$

and so $y = x$ and similarly $x = z$ so that $x = y = z$. Using the constraint we find

$$x = y = z = \frac{1}{3} \ .$$

The extremal value of f is therefore $1/3$.

It remains to decide whether this is a maximum or a minimum. Eliminating, e.g., the variable x using the constraint yields

$$f(1 - y - z, y, z) = (y + z)(1 - (y + z)) + yz = y + z - y^2 - z^2 - yz \ .$$

This can be made arbitrarily negative by choosing y and z large positive and hence there is no minimum. More can be said. Since

$$y + z - y^2 - z^2 - yz \leq y + z - y^2 - z^2 + |y||z| \leq y + z - y^2 - z^2 + \frac{1}{2}(x^2 + y^2) = y + z - \frac{1}{2}(x^2 + y^2)$$

we know that the as soon as y and z get large (positive or negative) the function becomes negative and hence the point we found must be a maximum.