1. Prepquiz 3 B

Problem 1: Compute the average

$$\frac{1}{A(R)}\int_R (x^2-y^2)dA$$

where the region R is given by the circle of radius r centered at the point (a,b) and $A(R)$ denotes its area.

The area of a circle of radius r is πr^2 no matter where the circle is located. It depends solely on the radius. Now we write

$$x = a+u \text{ and } y = b+v$$

and note that when (x,y) runs is inside the disk centered at (a,b) then (u,v) runs inside the disk D centered at the origin and conversely. Now imagine a small rectangle inside the disk centered at (a,b) given by

$$x_0-\delta \le x \le x_0+\delta \text{ and } y_0-\varepsilon \le y \le y_0+\varepsilon \ .$$

Obviously the rectangle is centered at (x_0,y_0) and has area $4\delta\varepsilon$. Now in the u,v variables we have that

$$x_0-a-\delta \le u \le x_0-a+\delta \text{ and } y_0-b-\varepsilon \le v \le y_0-b+\varepsilon \ .$$

which means that u,v range over a rectangle inside the disk centered at the origin and the area of this rectangle is also $4\delta\varepsilon$. The upshot of this is that the integral

$$\frac{1}{\pi r^2}\int_R (x^2-y^2)dxdy = \frac{1}{\pi r^2}\int_D ((a+v)^2-(b+v)^2)dudv = \frac{1}{\pi r^2}\int_D (a^2-b^2+2au-2bv+v^2-u^2)dudv \ .$$

Now we use polar coordinates $u=\rho\cos\theta, v=\rho\sin\theta$ and get for the last integral

$$\frac{1}{\pi r^2}\int_0^{2\pi}\left(\int_0^r (a^2-b^2+2a\rho\cos\theta-2b\rho\sin\theta+r^2(\cos^2\theta-\sin^2\theta))\rho d\rho\right)d\theta$$

which by interchanging the order of integration equals

$$\frac{1}{\pi r^2}\int_0^r\int_0^{2\pi}\left((a^2-b^2+2a\rho\cos\theta-2b\rho\sin\theta+r^2(\cos^2\theta-\sin^2\theta))d\theta\right)\rho d\rho = a^2-b^2$$

Problem 2: Compute the integral

$$\int_R x\sqrt{x^2+y^2}dA$$

where R is the piece of the unit disk in the first quadrant of the xy plane. Work this problem twice, once in cartesian coordinates and then again in polar coordinates.

The piece of the circle in the first quadrant is given by the equation $y = \sqrt{1-x^2}$ or $x = \sqrt{1-y^2}$. Now first integrating over y keeping x fixed and then integrating over x we get

$$\int_0^1 \left(\int_0^{\sqrt{1-x^2}} x\sqrt{x^2+y^2}dy \right) dx \ .$$

This integral can be solved with trigonometric substitution. But the factor x in front of the root suggests that it is better to integrate first with respect to x and then with respect to y and in this way we get

$$\int_0^1 \left(\int_0^{\sqrt{1-y^2}} x\sqrt{x^2+y^2}dx \right) dy \ .$$

An antiderivative of $x\sqrt{x^2+y^2}$ is $(x^2+y^2)^{3/2}/3$ and hence the above expression equals

$$\frac{1}{3}\int_0^1 (x^2+y^2)^{3/2}|_0^{\sqrt{1-y^2}} dy = \frac{1}{3}\int_0^1 [(1-y^2+y^2)^{3/2} - (y^2)^{3/2}]dy$$

which equals

$$\frac{1}{3}\int_0^1 [1-y^3]dy = \frac{1}{3}\frac{3}{4} = \frac{1}{4} \ .$$

In terms of polar coordinates we have that $0 \le \theta \le \pi/2$ and $0 \le r \le 1$ and we have to compute

$$\int_0^{\pi/2} \left(\int_0^1 (r^2 \cos\theta) r dr \right) d\theta = \int_0^{\pi/2} \cos\theta d\theta \int_0^1 r^3 dr = \frac{1}{4} \ .$$

Problem 3: (Taken from Grossman) Among all the ellipses

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

passing through the point $(3, 5)$ find the one that has smallest area.

The area of the ellipse is πab, $a > 0, b > 0$. We have minimize this function subject to the constraint

$$\left(\frac{3}{a}\right)^2 + \left(\frac{5}{b}\right)^2 = 1 \ .$$

Applying the theory of Lagrange multiplier yields

$$b = -\lambda\frac{3^2}{a^3} \text{ and } a = -\lambda\frac{5^2}{b^3}$$

together with the constraint. This means that

$$\frac{ba^3}{3^2} = -\lambda = \frac{ab^3}{5^2}$$

so that

$$\frac{a^2}{3^2} = \frac{b^2}{5^2} \ .$$

Hence

$$\left(\frac{3}{a}\right)^2 + \left(\frac{5}{b}\right)^2 = 2\left(\frac{3}{a}\right)^2 = 1$$

or

$$\frac{a^2}{3^2} = \frac{b^2}{5^2} = 2$$

and

$$a = 3\sqrt{2}\ , b = 5\sqrt{2}\ .$$