1. Solutions of Prepquiz 4A

Problem 1: To compute I_x we have to integrate the square of the distance to the x-axis over the region which means we have to compute

$$I_x = \int_0^a \int_0^b \int_0^c (y^2 + z^2) dz dy dx = \frac{acb^3 + abc^3}{3} = abc\frac{b^2 + c^2}{3} = M\frac{b^2 + c^2}{3} \ ,$$

where M is the mass. Likewise

$$I_y = M\frac{a^2 + c^2}{3} \ , \ I_z = M\frac{a^2 + b^2}{3} \ .$$

Problem 2: The cylinder in terms of cylindrical coordinates is given by $r = 2\cos\theta$ which restricts the possible values of θ to the intervals $[0, \pi/2]$ and $[3\pi/2, 2\pi]$. The sphere in terms of cylindrical coordinates is given by $r^2 + z^2 = 1$. To compute the volume of the region which is above the xy plane, inside the cylinder and below the sphere, we have to work out an integral of the type

$$\int\int\int dz r dr d\theta = \int\int \sqrt{1 - r^2} r dr d\theta$$

over the region in the xy plane which consists of the intersection of the disk of radius 1 and the disk centered at $(1, 0)$. This intersection is given by all those values of r and θ so that $0 \le r \le 1$ and $0 \le r \le 2\cos\theta$. Note that $2\cos\theta > 1$ for all $0 \le \theta < \pi/3$ and $5\pi/3 < \theta \le 2\pi$. For these angles r ranges over the interval $[0, 1]$. For the remaining angles in $[\pi/3, \pi/2]$ and $[3\pi/2, 5\pi/3]$ we have that $2\cos\theta \le 1$ and hence r ranges over the interval $[0, 2\cos\theta]$.

Hence we get the following contributions

$$\int_0^{\pi/3} \int_0^1 \sqrt{1 - r^2} r dr d\theta + \int_{\pi/3}^{\pi/2} \int_0^{2\cos\theta} \sqrt{1 - r^2} r dr d\theta$$

$$+ \int_{5\pi/3}^{2\pi} \int_0^1 \sqrt{1 - r^2} r dr d\theta + \int_{3\pi/2}^{5\pi/3} \int_0^{2\cos\theta} \sqrt{1 - r^2} r dr d\theta \ .$$

The last two integrals have the same value as the first two and hence we get

$$2\int_0^{\pi/3} \int_0^1 \sqrt{1 - r^2} r dr d\theta + 2\int_{\pi/3}^{\pi/2} \int_0^{2\cos\theta} \sqrt{1 - r^2} r dr d\theta \ .$$

Problem 3: Choose variables $u = x + 2y$ and $v = x - y$ and note that u ranges over the interval $[1, 2]$ and v over the interval $[2, 4]$. We have to compute the variables x, y in terms of u and v and get

$$y = \frac{1}{3}(u - v) \ , \ x = \frac{1}{3}(u + 2v)$$

so that the Jacobian determinant is given by

$$\det \begin{bmatrix} 1/3 & 2/3 \\ 1/3 & -1/3 \end{bmatrix} = -\frac{1}{3} \ .$$

The area is now given by

$$\int_1^2 \int_2^4 \frac{1}{3} dv du = \frac{2}{3} \ .$$