1. Solutions for Prepquiz 4B

Problem 1: Find the area of the region in the first quadrant of the xy plane given by $1 \le x^2 + 2y^2 \le 2$ and $1 \le x/y \le 2$.

Set $u = x^2 + 2y^2$ and $v = \frac{x}{y}$. Note that both variables, u as well as v vary between 1 and 2. The next step is to solve x, y in terms of u, v which is a bit trickier. We have that $x = yv$ and hence $u = y^2(v^2 + 2)$ so that

$$y = \frac{\sqrt{u}}{\sqrt{2+v^2}} \ , \ x = \frac{v\sqrt{u}}{\sqrt{2+v^2}} \ .$$

Now we compute

$$\frac{\partial x}{\partial u} = \frac{v}{2\sqrt{u}\sqrt{2+v^2}} \ , \ \frac{\partial x}{\partial v} = \frac{2\sqrt{u}}{(2+v^2)^{3/2}} \ , \ \frac{\partial y}{\partial u} = \frac{1}{2\sqrt{u}\sqrt{2+v^2}} \ , \ \frac{\partial y}{\partial v} = -\frac{\sqrt{u}v}{(2+v^2)^{3/2}} \ .$$

and find

$$|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}| = \left| -\frac{v^2}{2(2+v^2)^2} - \frac{1}{(2+v^2)^2} \right| = \frac{1}{2(2+v^2)} \ .$$

Hence we have to integrate

$$\int_1^2 \int_1^2 \frac{1}{2(2+v^2)} du dv = \int_1^2 \frac{1}{2(2+v^2)} dv = \frac{1}{4}\int_1^2 \frac{1}{1+v^2/2} dv \ .$$

Changing variables $x = v/\sqrt{2}$ we find that this integral equals

$$\frac{\sqrt{2}}{4}\int_{1/\sqrt{2}}^{\sqrt{2}} \frac{1}{1+x^2} dx = \frac{\sqrt{2}}{4}\left[\tan^{-1}(\sqrt{2}) - \tan^{-1}(1/\sqrt{2})\right] \ .$$

Problem 2: Find the volume of the region bounded below by the xy plane, bounded above by the surface $z = 3\sqrt{1 - x^2 - y^2}$ and laterally by the cylinder $r = \cos\theta$.

We set up the integral in cylindrical coordinates, $x = r\cos\theta, y = r\sin\theta, z$. First we note that $\cos\theta$ has to be non-negative which means that θ can take values in the intervals $[0, \pi/2]$ and $[3\pi/2, 2\pi]$. Further r must be in the interval $[0, \cos\theta]$. Hence we get

$$\int_0^{\pi/2}\int_0^{\cos\theta}\int_0^{3\sqrt{1-r^2}} dz r dr d\theta + \int_{3\pi/2}^{2\pi}\int_0^{\cos\theta}\int_0^{3\sqrt{1-r^2}} dz r dr d\theta$$

$$= 2\int_0^{\pi/2}\int_0^{\cos\theta} 3\sqrt{1-r^2} r dr d\theta = \int_0^{\pi/2}\int_0^{(\cos\theta)^2} 3\sqrt{1-u} du d\theta$$

$$= \frac{2}{3}3\int_0^{\pi/2}\left[1 - (1-(\cos\theta)^2)^{3/2}\right] d\theta = 2\int_0^{\pi/2}\left[1 - (\sin\theta)^3\right] d\theta$$

Problem 3: Find the center of mass of a body with constant mass density δ bounded below by the plane $z = 1$ and above by the surface $z = 4 - x^2 - y^2$.

Clearly, by symmetry, the x, y components of the center of mass both vanish. We have to compute two integrals, The total mass M and the the integral of the function z over the domain. It is easiest to do this integral using cylindrical coordinates. The variable z runs from 1 to $4 - r^2$, the variable r runs over all values so that $4 - r^2 \geq 1$, i.e., $0 \leq r \leq \sqrt{3}$. The mass is given by

$$M = \delta \int_0^{2\pi} \int_0^{\sqrt{3}} \int_1^{4-r^2} r dz dr d\theta = \delta \int_0^{2\pi} \int_0^{\sqrt{3}} (3 - r^2) r dr d\theta = \delta 2\pi \int_0^3 (3 - u) du = 9\pi\delta$$

Further

$$\delta \int_0^{2\pi} \int_0^{\sqrt{3}} \int_1^{4-r^2} r z dz dr d\theta = \delta \int_0^{2\pi} \int_0^{\sqrt{3}} r \frac{z^2}{2}\Big|_1^{4-r^2} dr d\theta = \delta\pi \int_0^{\sqrt{3}} r[(4 - r^2)^2 - 1] dr$$

$$= \delta\pi \int_0^3 [(4 - u)^2 - 1] du = \delta\pi[\frac{4^3 - 1}{3} - 3]$$

so that the z component of the Center of Mass is given by

$$\frac{4^3 - 10}{27} = 2 \ .$$