1. Quiz 3 solutions

Problem 1(10 points): Let $f(x, y) = xy$. Find the integral of the function f over the region bounded by $y = x(x-2)$ and $y = x$.

The line $y = x$ intersects the parabola $y = x(x-2)$ at the points 0 and 3. The curve $y = x$ bounds the region above and the curve $y = x(x-2)$ bounds it from below. Integrating first over the variable y and then over x we find for the integral

$$\int_0^3 \left(\int_{x(x-2)}^x xy dy \right) dx = \int_0^3 x \left(\int_{x(x-2)}^x y dy \right) dx = \frac{1}{2} \int_0^3 x(x^2 - x^2(x-2)^2) dx$$

$$= \frac{1}{2} \left(-\frac{3}{4} x^4 + \frac{4}{5} x^5 - \frac{1}{6} x^6 \right) \Big|_0^3 = \frac{3^5}{40} \ .$$

Problem 2: (10 points) Set up the $\int_R \int x dA$ for the region in the first quadrant, bounded above by the curve $x^2 + y^2 = 1$, and bounded below by $y = x$. Do this in cartesian coordinates (5 points) and polar coordinates (5 points).

In the domain R for fixed x the variable y ranges from x to $\sqrt{1-x^2}$ while the variable x ranges from 0 to $1/\sqrt{2}$. Hence, by first integrating with respect to y and then with respect to x we get

$$\int_0^{1/\sqrt{2}} \left(\int_x^{\sqrt{1-x^2}} x dy \right) dx \ .$$

To set up the integral in polar coordinates we have to describe the domain in terms of r and θ. The radius r ranges from 0 to 1 whereas the angle θ ranges from $\pi/4$ to $\pi/2$. The function x expressed in terms of polar coordinates is given by $r \cos\theta$. Hence the integral is

$$\int_{\pi/4}^{\pi/2} \left(\int_0^1 r \cos\theta r dr \right) d\theta \ .$$

You were not asked to evaluate the integral but its value is

$$\frac{1}{3}\left(1 - \frac{1}{\sqrt{2}}\right) \ .$$

Problem 3: Find the point (x, y) on the curve $y^2(1-x) - x^2(x+3) = 0$ where the x coordinate is minimum.

Using Lagrange multiplier with $f(x, y) = x$ and $g(x, y) = y^2(1-x) - x^2(x+3)$ the equation

$$\nabla f = \lambda \nabla g \text{ together with } g = 0$$

turns into

$$1 = \lambda(-y^2 - 3x^2 - 6x) \text{ and } 0 = 2y(1-x) \text{ together with } y^2(1-x) - x^2(x+3) = 0$$

The first serves only to determine λ which is not of interest. Hence we use only the second and third to solve for x, y,

$$2y(1-x) = 0 \text{ and } y^2(1-x) - x^2(x+3) = 0$$

Solving the second yields $x = 1$ and $y = 0$. The third equation, however, cannot be satisfied with $x = 1$. and therefore $y = 0$ and hence $x^2(x+3) = 0$ or $x = 0$. Which means that $x = 0$ or $x = -3$. Clearly $-3 < 0$. So the solution is the point $(-3, 0)$.

Another way of solving this problem is to assume that x is a function of y. Thus, at the point in question $x'(y) = 0$. Differentiating implicitly yields

$$2y(1-x) - 2yx' - 3x^2x' - 6xx' = 0$$

and since $x' = 0$ we find $2y(1-x) = 0$. The rest is as outlined above.

Extra Credit: What is the area of the region R in the (x, y) plane for which the integral

$$\int_R\int \left(9 - \frac{x^2}{4} - 4y^2\right) dA$$

is as large as possible.

The region R is given by all those points (x, y) where $9 - \frac{x^2}{4} - 4y^2 \geq 0$. R cannot have any parts where the integrant is negative (that would diminish the integral). Further if there is any domain not part of R where the integrant is positive we could add it to R and increase the integral. The region R is the region bounded by the ellipse

$$\frac{x^2}{4} + 4y^2 = 9 \text{ or } \frac{x^2}{6^2} + (\frac{2}{3})^2 y^2 = 1 \ .$$

The large semi axis is $a = 6$ and the small one is $b = 3/2$ and hence the area is $\pi ab = 9\pi$.