Quiz 4 Math 2401

Tom Morley & Michael Loss
Closed book, no notes, calculators allowed.

Problem 1 (10 points)

Find the z coordinate of the center of mass of a body with constant mass density δ bounded below by the plane z = 0 and above by the sphere of radius 2 with center at **0**.

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Volume is (1 / 2) (4 / 3) π r ^ 3
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volume = (2 / 3) π 2 ^ 3
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$$\frac{16\,\pi}{3}$$

$$\int_0^{\sqrt{2^\wedge 2 - r^\wedge 2}} z\, r\, dz$$

$$2\, r - \frac{r^3}{2}$$

$$\int_0^2 \int_0^{\sqrt{2^\wedge 2 - r^\wedge 2}} z\, r\, dz\, dr$$

$$2$$

$$\text{zMoment} = \int_0^{2\pi} \int_0^2 \int_0^{\sqrt{2^\wedge 2 - r^\wedge 2}} z\, r\, dz\, dr\, d\theta$$

$$4\,\pi$$

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zBar = zMoment / volume
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$$\frac{3}{4}$$

Can also be done in spherical coordinates

$$\int_0^{\pi/2} \int_0^{2\pi} \int_0^2 \rho\, \text{Cos}[\phi]\, \rho^\wedge 2\, \text{Sin}[\phi]\, d\rho\, d\theta\, d\phi$$

$$4\,\pi$$

Again divide by volume

8 points for correcly setting up the two integrals. 2 points for evaluation of integrals. Note that the volume is 1/2 a sphere, and need not be set up as an integral. So getting demoninator as (1/2) (4/3) π r^3 (or since r = 2, (4/3) π 2^3) is 5 points.

Problem 2 (10 points)

Compute the volume of the region in the octant x⩾0, y⩽0, z⩾0, which is inside the cylinder $x^2 + y^2 = 1$, above the xy plane, and below the surface $z = 4 + x^2 - y^2$. Just fill in the missing limits and integrand. Do Not Evaluate!

$$\int \quad \int_0^1 \quad \int_0 \quad dz\,dr\,d\theta$$

In[1]:= $$\int_{-\pi/2}^{0}\int_0^1\int_0^{4 + r^2 (\cos[\theta]^2 - \sin[\theta]^2)} r\,dz\,dr\,d\theta$$

Out[1]= π

or

In[2]:= $$\int_{3\pi/2}^{2\pi}\int_0^1\int_0^{4 + r^2 (\cos[\theta]^2 - \sin[\theta]^2)} r\,dz\,dr\,d\theta$$

Out[2]= π

Note : Does not ask for evaluation. Typical grading: 7 points for θ limits wrong. 8 points for all but upper z wrong. 5 points for almost anything.

Any reasonable response should be at least 5. Do not take off multiple times if the limits are in the wrong order, etc. Note: you can simplify (Cos[θ]^2 - Sin[θ]^2) by trig identities

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In[3]:= Simplify[(Cos[θ]^2 - Sin[θ]^2)]
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Out[3]= Cos[2 θ]

Problem 3 (10 points)

Let u= xy and v = x/y. Find x and y in terms of u and v. Use this to find the area of the region in the first quadrant of the xy plane given by $1 \le xy \le 2$ and $2 \le x/y \le 4$. Hint: Change variables.

```
Clear[u, v, x, y]

ans = Solve[{ u == x y, v == x / y}, {x, y}]
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$$\left\{\left\{x \to -\sqrt{u}\,\sqrt{v},\ y \to -\frac{\sqrt{u}}{\sqrt{v}}\right\},\ \left\{x \to \sqrt{u}\,\sqrt{v},\ y \to \frac{\sqrt{u}}{\sqrt{v}}\right\}\right\}$$

```
old = {x, y} /. ans[[2]]
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$$\left\{\sqrt{u}\,\sqrt{v},\ \frac{\sqrt{u}}{\sqrt{v}}\right\}$$

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(matrix = {D[old, u], D[old, v]}) // MatrixForm
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$$\begin{pmatrix} \frac{\sqrt{v}}{2\sqrt{u}} & \frac{1}{2\sqrt{u}\sqrt{v}} \\ \frac{\sqrt{u}}{2\sqrt{v}} & -\frac{\sqrt{u}}{2 v^{3/2}} \end{pmatrix}$$

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Det[matrix]
```

$$-\frac{1}{2 v}$$

With absolute value, it is 1/(2 v)

Now just integrate :

$$\int_2^4 \int_1^2 1/(2 v)\, du\, dv$$

$$\frac{\text{Log}[2]}{2}$$

4 points solve for x y
4 points for Jacobian 1/(2 v)
2 points for integral and evaluation

The other order gives almost the same and is correct.
Note: Asks for evaluation, but this should only be a point or so.

Extra Credit : (5 points)

Find the volume of the region inside the sphere $x^2 + y^2 + z^2 = 25$, and outside the cylinder $x^2 + y^2 = 9$.

Answer is

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(4 / 3) π (Sqrt[25 - 9] ^ 3)
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$$\frac{256\,\pi}{3}$$

Unless they get this, not much partial credit.
3 points for correctly setting up integral .
Only get 5 if the integral is correctly evaluated.