SOLUTIONS FOR HOMEWORK 1

Problem 1: a) We have the inequalities

$$\max_{1\le j\le d}|x_j| \le \|x\|_p \le \max_{1\le j\le d}|x_j| d^{1/p}$$

As $p\to\infty$, $d^{1/p}\to 1$.

b) We have to show that

$$\|f\|_{L_q} \le \|f\|_{L_p}$$

which implies that whenever $f\in L_p$ we have that $f\in L_q$, i.e., $L_q\subset L_p$. We use Hölder's inequality. Assume first that f is bounded. Write

$$\int_a^b |f(x)|^q dx = \int_a^b 1\cdot |f(x)|^q dx \le \left(\int_a^b 1^r dx\right)^{1/r} \left(\int_a^b [|f(x)|^q]^s dx\right)^{1/s}$$

where $1/r+1/s=1$. Now we pick $s=p/q\ge 1, r=$ and we get

$$\left(\int_a^b |f(x)|^q dx\right)^{1/q} \le ([b-a])^{\frac{p-q}{pq}} \left(\int_a^b |f(x)|^p dx\right)^{1/p} .$$

By monotone convergence, we now can remove the condition that f is bounded.

Problem 2: a) Assume that B is closed and $x_n\in B$ a sequence that converges to $x\in X$. We have to show that $x\in B$. Suppose not, i.e., x is in the complement of B, which by assumption is open. Hence there exists a ball centered at x with positive radius which is entirely contained in the complement of B. Since $x_n\in B$ converges to x there must be points of the sequence in the ball, which is a contradiction. Conversely, assume that B is a set with the property that for every convergent sequence in B the limit is also in B. We have to show that the complement of B is open. Suppose not. Hence, there exists a point x in the complement of B such that every ball centered at x with sufficiently small radius intersects B. Pick such a ball C_1 which contains some $x_1\in B$. Since $x_1\ne x$ we can find a smaller ball C_2 that does not contain x_1 but another point $x_2\in B$. Continuing in this way, we can construct a sequence of points that converge to x and hence $x\in B$, a contradiction.

b) Consider any collection of open set $U_\iota, \iota\in I$ and form their union $U=\cup_{\iota\in I}U_\iota$. Pick any point $x\in U$. Then $x\in U_\iota$ for some $\iota\in I$. Since U_ι is open there exists an open ball centered at x which is contained in U_ι and hence in U. Thus, U is open. The statement about intersections follows from $\cap_{\iota\in I}C_\iota = [\cup_{\iota\in I}C_\iota^{c}]^{c}$ where C^{c} is the complement of C in the space X.

Problem 3: It suffices to show that an arbitrary norm $\|\cdot\|$ on $\mathbb{R}^d$ is equivalent to the Euclidean norm $|\cdot|$. We start with

$$\|x\| = \|\sum_{j=1}^d x_j e_j\| \le \sum_{j=1}^d |x_j|\|e_j\| \le \left(\sum_{j=1}^d \|e_j\|^2\right)^{1/2} |x| \ .$$

The lower bound is a bit trickier. We have to find a positive lower bound on the ratio

$$\frac{\|x\|}{|x|} = \|\frac{x}{|x|}\|$$

in other words we have to find a positive lower bound on the function $\|x\|$ restricted to the Euclidean sphere of radius 1. The sphere is compact and $\|x\|$ is a continuous function on the sphere and hence attains its minimum at some point x_0 with $|x_0| = 1$. Thus $\|x_0\| > 0$ and

$$\|x\| \ge \|x_0\||x| \ .$$

Problem 4: a) Let $x^n \in \ell_p$ be a C.S., i.e, for $\varepsilon > 0$ arbitrary there exists N so that

$$\left(\sum_{j=1}^\infty |x_j^n - x_j^m|^p\right)^{1/p} < \varepsilon$$

for all $m, n > N$. Hence x_j^n is a Cauchy sequence and hence has a limit x_j. Now

$$\left(\sum_{j=1}^K |x_j - x_j^m|^p\right)^{1/p} = \lim_{n\to\infty} \left(\sum_{j=1}^K |x_j^n - x_j^m|^p\right)^{1/p} < \varepsilon$$

for any K and all $m > N$. Since the right side does not depend on K we may let $k \to \infty$ and get that

$$\|x - x^m\| = \left(\sum_{j=1}^\infty |x_j - x_j^m|^p\right)^{1/p} \le \varepsilon$$

for all $m > N$. Because $|\|x\| - \|x^m\|| \le \|x - x^m\|$, $x \in \ell_p$. Thus, x^m converges to x.

b) The case ℓ_∞ is similar. Each x_j^n converges to some x_j. For every j we have that

$$|x_j - x_j^m| = \lim n \to \infty |x_j^n - x_j^m| < \varepsilon$$

all $m > N$, and hence it follows that

$$\|x - x^m\|_\infty = \sup_j |x_j - x_j^m| \le \varepsilon \ .$$

This implies that $x \in \ell_\infty$ and that x^m converges to x in ℓ_∞.