HOMEWORK 2, DUE ON TUESDAY SEPTEMBER 27.

Problem 1: Show that $\ell_1^* = \ell_\infty$.

First we show that $\ell_\infty \subset \ell_1^*$. Pick $a \in \ell_\infty$ and consider the linear functional on ℓ_1 given by

$$f_a(b) = \sum_{j=1}^{\infty} a_i b_i$$

then

$$|f_a(b)| \le \sup_j |a_i| \sum_{j=1}^{\infty} |b_j| = \|a\|_{\ell_\infty} \|b\|_{\ell_1}$$

and hence $f_a \in \ell_1^*$. Conversely, if $f \in \ell_1^*$ and e_j are the canonical vectors, then

$$f(b) = \sum_{j=1}^{\infty} f(e_j) b_j \ .$$

This follows since the sequence b_j is summable. Moreover, $|f(e_j)| \le \|f\|_{\ell_1^*} \|e_j\|_{\ell_1} = \|f\|_{\ell_1^*}$. Hence $\{f(e_j)\} \in \ell_\infty$.

Problem 2: Prove that any finite dimensional normed space is reflexive.

Set $N = \dim X$. We know from the lecture that for any basis in $e_1, \dots, e_N$ there exists a basis $f_1, \dots, f_N \in X^*$ such that

$$f_i(e_j) = \delta_{i,j} \ .$$

Hence, $\dim X^* = N$. The same argument yields $\dim X^{**} = N$. Because $X \subset X^{**}$ is a subspace and X has the same dimension as X^{**} we have that $X = X^{**}$.

Problem 3: Show that ℓ_∞ is not separable. (Hint: Consider balls of small radii centered at sequences with integer coefficients. Show that there are uncountably many such balls.)

Pick the any subset $A \subset \mathbb{N}$ and define vectors $e_A \in \ell_\infty$ by setting $e_j = 1$ if $j \in A$ and $e_j = 0$ if $j \notin A$. There are uncountably many such vectors. For any such two vectors e_A, e_B with $A \ne B$ we have that

$$\|e_A - e_B\| = 1 \ .$$

Now consider balls of radius $1/3$ centered the vectors e_A with $A \subset \mathbb{N}$. These are uncountably many disjoint balls in ℓ_∞ and hence this space is not separable.

Problem 4: Prove that if $\{\phi_n(x)\}_{n=1}^\infty$ is an orthonormal basis in $L^2[a,b]$ then for all $x \in [a,b]$

$$\sum_{n=1}^\infty |\int_a^x \phi_n(t)dt|^2 = x - a \ .$$

(The converse also holds, but is a bit trickier to prove.)

Write as an inner product

$$\int_a^x \phi_n(t)dt = \langle \phi_n, \chi_a \rangle$$

where χ_a is the characteristic function of the set $[a,x]$. Then, by the Parseval formula

$$\sum_{n=1}^\infty |\langle \phi_n, \chi_a \rangle|^2 = \int |\chi_a(t)|^2 dt = \int_a^x dt = (x-a) \ .$$

Problem 5: Let X be a normed space and Y be a Banach Space. Show that the space of linear bounded operators $L(X \mapsto Y)$ is a Banach space.

Let A_n be a Cauchy sequence in $L(X \mapsto Y)$. Then the sequence $\|A_n\|$ is uniformly bounded. To see this, pick $\varepsilon > 0$ and write

$$\|A_n\| \le \|A_m\| + \|A_n - A_m\| \ .$$

There exists N so that for all $n, m > N$ the second term on the right side is less than ε. Fixing $m > N$ shows that $\|A_n\|$ is bounded independent of n. Pick any $x \in X$. Then $A_n x \in Y$ is a Cauchy sequence in Y and since Y is complete it converges to some vector which we denote by Ax. It is plain that the map $x \to Ax$ is linear. Moreover,

$$\|Ax\| = \lim_{n\to\infty} \|A_n x\| \le \sup_n \|A_n\| \|x\|$$

and hence A is a bounded operator. Finally for $m > N$,

$$\|Ax - A_m x\| = \lim_{n\to\infty} \|A_n x - A_m x\| < \varepsilon \|x\| \ ,$$

i.e.,

$$\|A - A_m\| < \varepsilon \ .$$

Hence A_n converges to A in norm.