HOMEWORK 3, DUE THURSDAY OCTOBER 13

Problem 1: (10 points) Let X be a metric space. We have shown in class that a set $B \subset X$ is relatively compact if and only if for any $\varepsilon > 0$ there exists a finite ε-net for B, which is not necessarily in B. Show that if B is relatively compact then for any $\varepsilon >$ there exists a finite ε-net for B which is in B.

Problem 2: (10 points) Let X is a Banach space and $K : X \to X$ a compact operator. We know by definition that if x_n is a bounded sequence then Kx_n has a convergent subsequence. If we denote this limit by y one might think that y must be in the range of K, i.e., there exist $x \in X$ such that $Kx = y$. Show, by example that generally this is not true.

Hint: Consider $X = C[-1, 1]$ and K to be the operator

$$Kx(t) = \int_{-1}^{t} x(t)dt$$

where $x(t) \in C[-1, 1]$. We know from the lecture that this operator is compact. Consider the sequence $x_n(t)$ where

$$x_n(t) = \begin{cases} 0 \ , & -1 \le t \le 0 \\ nt \ , & 0 \le t \le 1/n \\ 1 \ , & 1/n \le t \le 1 \ . \end{cases}$$

Problem 3: (15 points) On ℓ_2 consider the operator

$$Tx = (0, x_1, \frac{x_2}{2}, \dots, \frac{x_n}{n}, \dots) \ .$$

a) Show that T is compact.
b) Find $\sigma_p(T)$.
c) Find $\sigma_r(T)$.

Problem 4: (15 points) Let $S \subset \mathbb{C}$ be a compact subset of the complex numbers. Find a bounded linear operator $A : \ell_2 \to \ell_2$ such that $\sigma(A) = S$.