HOMEWORK 3, DUE THURSDAY OCTOBER 13

Problem 1: (10 points) Let X be a metric space. We have shown in class that a set $B \subset X$ is relatively compact if and only if for any $\varepsilon > 0$ there exists a finite ε-net for B, which is not necessarily in B. Show that if B is relatively compact then for any $\varepsilon >$ there exists a finite ε-net for B which is in B.

Pick any ε. There exists a finite $\varepsilon/2$-net for the set B, i.e., there exist finitely many points $N = \{x_1, \ldots, x_K\} \subset X$ such that for any $x \in B$ there exists $x_j \in N$ with $d(x, x_j) < \varepsilon/2$. Conversely, for every $x_j \in N$ there exists a point $x \in B$ such that $d(x, x_j) < \varepsilon/2$, because otherwise, we may discard x_j because $N \setminus \{x_j\}$ is still an $\varepsilon/2$ net. Thus, for each x_j we may find a point $y_j \in B$ such that $d(x_j, y_j) < \varepsilon/2$. Thus, if $x \in B$ there exists y_j so that

$$d(x, y_j) \leq d(x, x_j) + d(x_j, y_j) < \varepsilon$$

and the set $\{y_1, \ldots, y_K\} \subset B$ is a finite ε net for B.

Problem 2: (10 points) Let X is a Banach space and $K : X \to X$ a compact operator. We know by definition that if x_n is a bounded sequence then Kx_n has a convergent subsequence. If we denote this limit by y one might think that y must be in the range of K, i.e., there exist $x \in X$ such that $Kx = y$. Show, by example that generally this is not true.

Hint: Consider $X = C[-1, 1]$ and K to be the operator

$$Kx(t) = \int_{-1}^{t} x(t)dt$$

where $x(t) \in C[-1, 1]$. We know from the lecture that this operator is compact. Consider the sequence $x_n(t)$ where

$$x_n(t) = \begin{cases} 0\ , & -1 \leq t \leq 0 \\ nt\ , & 0 \leq t \leq 1/n \\ 1\ , & 1/n \leq t \leq 1\ . \end{cases}$$

Note that

$$Kx_n(t) = \begin{cases} 0\ , & -1 \leq t \leq 0 \\ nt^2/2\ , & 0 \leq t \leq 1/n \\ t - \frac{1}{2n}\ , & 1/n \leq t \leq 1\ . \end{cases}$$

As $n \to \infty$ we find that Kx_n converges uniformly to the function

$$y(t) = \begin{cases} 0\ , & -1 \leq t \leq 0 \\ t\ , & 0 < t \leq 1\ . \end{cases}$$

There is, however, no function $x \in C[-1, 1]$ with the property that $Kx = y$, because y is not continuously differentiable.

Problem 3: (15 points) On ℓ_2 consider the operator

$$Tx = (0, x_1, \frac{x_2}{2}, \dots, \frac{x_n}{n}, \dots) .$$

a) Show that T is compact.
b) Find $\sigma_p(T)$.
c) Find $\sigma_r(T)$.

a) Define the finite rank operator $T_N : \ell_2 \to \ell_2$ by

$$T_N x = (0, x_1, \frac{x_2}{2}, \dots, \frac{x_N}{N}, 0, 0 \dots)$$

and note that

$$\|Tx - T_N x\|^2 = \sum_{n=N+1}^{\infty} \frac{x_n^2}{(N+1)^2} \le \frac{1}{(N+1)^2}\|x\|^2$$

so that

$$\|T - T_N\| \le \frac{1}{(N+1)} .$$

Thus, T can be approximated to any accuracy by the compact operators T_N and hence T is compact.

b) There are no eigenvalues.

c) Because T is compact, the only possible point in $\sigma_r(T)$ is the 0. The range $\text{Ran}(T)$ consists of all vectors in ℓ_2 that have a 0 as the first entry. Hence, $\text{Ran}(T)$ is not dense and $0 \in \sigma_r(T)$.

Problem 4: (15 points) Let $S \subset \mathbb{C}$ be a compact subset of the complex numbers. Find a bounded linear operator $A : \ell_2 \to \ell_2$ such that $\sigma(A) = S$.

Pick any countable dense set of complex numbers in S. Such a set exists, e.g., the complex numbers in S with rational real and imaginary parts. Denote this set by $\{\lambda_1, \lambda_2, \lambda_3, \dots\}$. Now define the bounded operator A by

$$Ae_j = \lambda_j e_j$$

where e_j is the canonical basis in ℓ_2. Clearly every λ_j is an eigenvalue. These eigenvalues are dense in S and since the spectrum is closed $S \subset \sigma(A)$. Now pick any $\lambda \notin S$. There exist $\varepsilon > 0$ so that $|z - \lambda| > \varepsilon$ for all $z \in S$. Hence

$$(A - \lambda I)^{-1}x = ((\lambda_1 - \lambda)^{-1}x_1, (\lambda_2 - \lambda)^{-1}x_2, \dots)$$

and, moreover,

$$\|(A - \lambda I)^{-1}x\| \le \frac{1}{\varepsilon}\|x\|$$

and it follows that $\lambda \in \rho(A)$, i.e., the complement of S is a subset of the complement of $\sigma(A)$ and hence $\sigma(A) \subset S$.