HOMEWORK 4, DUE THURSDAY OCTOBER 27

Problem 1: (10 points) Let X be a Banach space, $A : X \to X$ be a bounded linear invertible with a bounded inverse and $K : X \to X$ a linear compact operator. Show that for any $y \in X$ the equation $Ax + Kx = y$ has a unique solution if and only if $Ax + Kx = 0$ has only the trivial solution.

Problem 2: (10 points) On the space $L^2[0, 1]$ find the spectrum of the operator

$$Kf(t) = \int_0^t f(s)ds \ .$$

Problem 3: (15 points) a) For the operator K in Problem 2, show that the Neumann Series, i.e., $\sum_{n=0}^{\infty} K^n$ exists.

b) Find a simple expression for

$$\sum_{n=0}^{\infty} K^n f(t)$$

where $f \in L^2[0, 1]$.

Hint: Compute $K^2 f, K^3 f$ and simplify using integration by parts. Then guess the general term and proceed by induction.

Problem 4: (15 points) Consider the operator $K : L^2[0, 1] \to L^2[0, 1]$ given by

$$Kf(t) = \int_0^1 \min\{t, s\} f(s)ds \ .$$

a) Prove that K is compact and self-adjoint.

b) Find the spectrum of K.

c) Find $\|K\|$.

Hint: Differentiate!