HOMEWORK 4, DUE THURSDAY OCTOBER 27

Problem 1: (10 points) Let X be a Banach space, $A : X \to X$ be a bounded linear invertible with a bounded inverse and $K : X \to X$ a linear compact operator. Show that for any $y \in X$ the equation $Ax + Kx = y$ has a unique solution if and only if $Ax + Kx = 0$ has only the trivial solution.

Because A has a bounded inverse we may write

$$A + K = A(I + A^{-1}K)$$

and note that $(A + K)x = y$ has a unique solution if and only if $(I + A^{-1}K)x = A^{-1}y$ has a unique solution. Since $A^{-1}K$ is compact we may apply the Fredholm alternative and concluded that this equation has a unique solution for any y if and only if $((I + A^{-1}K)x = 0$ has only the trivial solution which holds if and only if $Ax + Kx = 0$ has only the trivial solution.

Problem 2: (10 points) On the space $L^2[0, 1]$ find the spectrum of the operator

$$Kf(t) = \int_0^t f(s)ds \ .$$

The operator is a compact operator on $L^2[0, 1]$ and hence $0 \in \sigma(K)$. Thus, we only have to investigate the eigenvalues $\lambda \neq 0$.

$$\int_0^t f(s)ds = \lambda f(t) \ .$$

Since the left side is continuous, so is the right side, i.e., the function f is a continuous function. Thus, the left side is differentiable and hence f is also differentiable. (This is proving regularity by bootstrap). Thus, we may differentiate and get

$$f(t) = \lambda f'(t)$$

and hence

$$f(t) = ce^{t/\lambda}$$

where c is some constant. However, as $t \to 0$ the left side of the eigenvalue equation vanishes and hence $f(0) = 0$. Thus, $c = 0$ and the solution vanishes identically. Thus, there re no non-zero eigenvalues. So $\sigma(K) = \{0\}$.

Problem 3: (15 points) a) For the operator K in Problem 2, show that the Neumann Series, i.e., $\sum_{n=0}^{\infty} K^n$ exists.

b) Find a simple expression for

$$\sum_{n=0}^{\infty} K^n f(t)$$

where $f \in L^2[0,1]$.

Hint: Compute $K^2 f, K^3 f$ and simplify using integration by parts. Then guess the general term and proceed by induction.

we compute a bound on the norm of K.

$$\int_0^1 |Kf(t)|^2 dt = \int_0^1 | \int_0^t f(s) ds|^2 dt = \int_0^1 | \int_0^t 1 \cdot f(s) ds|^2 dt$$
$$\leq \int_0^1 [\int_0^t 1 ds][\int_0^t |f(s)|^2 ds] dt \leq \frac{1}{2} \|f\|^2_{L^2[0,1]} \ .$$

Hence, $\|K\| \leq \frac{1}{\sqrt{2}}$ and the Neumann series converges.

For b) note that, using integration by parts,

$$\int_0^t K^{n-1} f(s) ds = \int_0^t 1 \cdot K^{n-1} f(s) ds = s K^{n-1} f(s)|_0^t - \int_0^t s \frac{d}{ds} K^{n-1} f(s) ds = \int_0^t (t-s) K^{n-2} f(s) ds \ .$$

Another integration by parts yields

$$\int_0^t K^{n-1} f(s) ds = \int_0^t \frac{(t-s)^2}{2} K^{n-3} f(s) ds$$

and continuing this way we find that

$$K^n f(t) = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s) ds$$

Summing over n yields

$$\sum_{n=0}^{\infty} K^n f(t) = f(t) + \int_0^t e^{t-s} f(s) ds \ .$$

Remark: Note that the the properties of K are analogous to Nilpotent matrices, i.e., strictly upper or strictly lower triangular matrices.

Problem 4: (15 points) Consider the operator $K : L^2[0,1] \to L^2[0,1]$ given by

$$Kf(t) = \int_0^1 \min\{t,s\} f(s) ds \ .$$

a) Prove that K is compact and self-adjoint.

b) Find the spectrum of K.

c) Find $\|K\|$.

Hint: Differentiate!

The function $\min\{t,s\} \in C([0,1]\times[0,1])$ and hence the operator is compact and since

$$\overline{\min\{s,t\}} = \min\{t,s\}$$

the operator is self-adjoint.

Next we have to compute the eigenvalues.

Write

$$Kf(t) = \int_0^t sf(s)ds + t\int_t^1 f(s)ds = \lambda f(t) \tag{1}$$

Again, the left side of the equation is continuous and hence so is the right side. Feeding this information back to the left side shows that it is in fact differentiable and hence so is f. Differentiating yields

$$\int_t^1 f(s)ds = \lambda f'(t) \tag{2}$$

and once more,

$$-f(t) = \lambda f''(t)\ .$$

Solving this equation yields

$$f(t) = Ae^{kt} + Be^{-kt}$$

where $k^2 = -\frac{1}{\lambda}$. Note that k cannot be zero. Note that (2) requires that $f'(1) = 0$ and (1) requires that $f(0) = 0$. Hence

$$A + B = 0\ , k(Ae^k - Be^{-k}) = 0\ .$$

Solving this yields

$$kA\cosh k = 0, B = -A\ .$$

We must have nontrivial solutions and hence $\cosh k = 0$. Writing $k = i\kappa$ we must choose κ such that $\cos\kappa = 0$, i.e.,

$$\kappa = \frac{\pi}{2} + \pi k$$

where k is any integer. Hence, the eigenvalues are

$$\lambda_k = \frac{1}{(\frac{\pi}{2} + \pi k)^2}$$

and the eigenfunctions are

$$f_k(t) = A\sin(t[\frac{\pi}{2} + \pi k])\ .$$

For the norm

$$\|K\| = \max_k |\lambda_k| = \frac{4}{\pi^2}\ .$$