HOMEWORK 5, DUE TUESDAY NOVEMBER 14

Problem 1: (10 points) Let $B : H \to H$ be a bounded operator. Show that for any $t \in \mathbb{R}$, the exponential series

$$Exp(tB) := \sum_{j=0}^{\infty} \frac{t^j}{j!} B^j$$

converges in the operator norm.

Problem 2: (10 points) Show that for any $s, t \in \mathbb{R}$,

$$Exp(tB)Exp(sB) = Exp((s+t)B) \ .$$

Hint: Approximate the factors by finite sums, use the binomial formula and estimate.

Problem 3: (10 points) Let $A : H \to H$ be a bounded self-adjoint operator. Show that for any $t \in \mathbb{R}$ the operator

$$U = Exp(iAt)$$

is unitary, i.e., U is invertible and $U^*U = I$.

Problem 4: (10 points) Let $A : H \to H$ be a bounded, non-negative self-adjoint operator, i.e., $\langle Ax, x \rangle \geq 0$. Show that $(A + \lambda)^{-1}$ exist and is a bounded operator for all $\lambda > 0$, i.e., $\lambda \in \rho(A)$, the resolvent set of A.

Problem 5: (10 points) Let A and B be two bounded positive self-adjoint operators, both with bounded inverses. Assume that $A < B$. Prove that

$$B^{-1} < A^{-1} \ .$$