HOMEWORK 5, DUE TUESDAY NOVEMBER 14

Problem 1: (10 points) Let $B : H \to H$ be a bounded operator. Show that for any $t \in \mathbb{R}$, the exponential series

$$Exp(tB) := \sum_{j=0}^{\infty} \frac{t^j}{j!} B^j$$

converges in the operator norm.

The sequence

$$A_n = \sum_{j=0}^{n} \frac{t^j}{j!} B^j$$

is a Cauchy sequence because for $n \geq m$

$$\|A_n - A_m\| \leq \sum_{j=m+1}^{n} \frac{t^j}{j!} \|B\|^j \leq \sum_{j=m+1}^{\infty} \frac{t^j}{j!} \|B\|^j \to 0$$

as $m \to \infty$. Since $L(H)$ is complete the sequence converges.

Problem 2: (10 points) Show that for any $s, t \in \mathbb{R}$,

$$Exp(tB)Exp(sB) = Exp((s+t)B) \ .$$

Hint: Approximate the factors by finite sums, use the binomial formula and estimate.

$$[\sum_{m=0}^{N} \frac{t^m}{m!} B^m][\sum_{n=0}^{N} \frac{s^n}{n!} B^n] = \sum_{k=0}^{2N} \frac{1}{k!} B^k \sum_{n=0}^{\min(k,N)} s^n t^{k-n} \frac{k!}{n!(k-n)!}$$

We write,

$$\sum_{k=0}^{2N} \frac{1}{k!} B^k \sum_{n=0}^{\min(k,N)} s^n t^{k-n} \frac{k!}{n!(k-n)!}$$

$$= \sum_{k=0}^{N} \frac{1}{k!} B^k \sum_{n=0}^{k} s^n t^{k-n} \frac{k!}{n!(k-n)!} + \sum_{k=N+1}^{2N} \frac{1}{k!} B^k \sum_{n=0}^{N} s^n t^{k-n} \frac{k!}{n!(k-n)!}$$

$$= \sum_{k=0}^{N} \frac{1}{k!} B^k (s+t)^k + R_N$$

where

$$R_N = \sum_{k=N+1}^{2N} \frac{1}{k!} B^k \sum_{n=0}^{N} s^n t^{k-n} \frac{k!}{n!(k-n)!}$$

Now,

$$\sum_{n=0}^{N} |s|^n |t|^{k-n} \frac{k!}{n!(k-n)!} \leq \sum_{n=0}^{k} |s|^n |t|^{k-n} \frac{k!}{n!(k-n)!} = (|s| + |t|)^k$$

and hence

$$\|R_N\| \leq \sum_{k=N+1}^{2N} \frac{1}{k!} \|B\|^k (|s| + |t|)^k$$

which tends to 0 as $N \to \infty$.

Problem 3: (10 points) Let $A : H \to H$ be a bounded self-adjoint operator. Show that for any $t \in \mathbb{R}$ the operator

$$U = Exp(iAt)$$

is unitary, i.e., U is invertible and $U^*U = I$.

This follows from the previous problem by noting that U is invertible since $U^{-1} = Exp(-iAt)$. Next, because

$$U = \sum_{j=0}^{\infty} \frac{(it)^j}{j!} A^j$$

we see that

$$U^* = \sum_{j=0}^{\infty} \frac{(-it)^j}{j!} A^{*j} = \sum_{j=0}^{\infty} \frac{(-it)^j}{j!} A^j = Exp(-iAt) \ .$$

Problem 4: (10 points) Let $A : H \to H$ be a bounded, non-negative self-adjoint operator, i.e., $\langle Ax, x \rangle \geq 0$. Show that $(A + \lambda)^{-1}$ exist and is a bounded operator for all $\lambda > 0$, i.e., $\lambda \in \rho(A)$, the resolvent set of A.

We have that

$$\|(A + \lambda I)x\|^2 = \|Ax\|^2 + 2\lambda \langle Ax, x \rangle + \lambda^2 \|x\|^2 \geq \lambda^2 \|x\|^2$$

since $\lambda > 0$ and $\langle Ax, x \rangle \geq 0$. Hence

$$\|(A + \lambda I)x\| \geq \lambda \|x\| \ . \tag{1}$$

This shows that $A + \lambda I$ is injective. Moreover, suppose that $y_n \in \mathrm{Ran}(A + \lambda I)$ is a sequence that converges to y in H. There exists x_n such that $y_n = (A + \lambda I)x_n$ and from (1) we see that for any $n \geq m$

$$\|y_n - y_m\| = \|(A + \lambda I)(x_n - x_m)\| \geq \lambda \|x_n - x_m\|$$

which shows that x_n is a Cauchy sequence which converges to some $x \in H$. Since A is bounded Ax_n converges to Ax and hence we have that

$$y = \lim_{n \to \infty} (A + \lambda I)x_n = (A + \lambda I)x$$

and hence $y \in \mathrm{Ran}(A + \lambda I)$, i.e., $\mathrm{Ran}(A + \lambda I)$ is closed. Because $A = A^*$

$$H = \overline{\mathrm{Ran}(A + \lambda I)} \oplus \mathrm{Ker}(A + \lambda I) = \mathrm{Ran}(A + \lambda I) \ .$$

Hence $(A + \lambda I)$ is invertible and from (1) we find that

$$\|(A + \lambda I)^{-1}\| \leq \frac{1}{\lambda} \ .$$

Another approach is to use the fact that the projection E_μ vanishes for $\mu < 0$ and hence $-\lambda$ is a regular point, i.e., $(A - (-\lambda)I) = (A + \lambda I)$ has a bounded inverse.

Problem 5: (10 points) Let A and B be two bounded positive self-adjoint operators, both with bounded inverses. Assume that $A < B$. Prove that

$$B^{-1} < A^{-1} \ .$$

First we prove the statement for the special case that $A = I$, i.e., that $B > I$ implies that $B^{-1} < I$. This follows from

$$\langle x, x \rangle = \langle BB^{-1/2}x, B^{-1/2}x \rangle > \langle B^{-1/2}x, B^{-1/2}x \rangle = \langle B^{-1}x, x \rangle$$

The statement $A < B$ is equivalent to $B^{-1/2}AB^{-1/2} < I$. By the previous argument

$$I < (B^{-1/2}AB^{-1/2})^{-1} = B^{1/2}A^{-1}B^{1/2}$$

which in turn is equivalent to

$$B^{-1} < A^{-1} \ .$$

Remark 0.1. *One can, quite generally, pose the question for which functions f is it true that $A \leq B$ implies $f(A) \leq f(B)$. This can be answered completely, namely all function of the form*

$$f(x) = \int_0^\infty \frac{1}{x + t} \mu(dt)$$

where μ is a positive measure. Note that it is false in general that $A \leq B$ implies that $A^2 \leq B^2$ for positive self adjoint operators.