HOMEWORK 1: DUE DATE TUESDAY SEPT. 13

Problem 1: Consider $\mathbb{R}^d$ with the norm

$$\|x\|_p = \left(\sum_{j=1}^{d} |x_j|^p\right)^{1/p}$$

a) Show that

$$\lim_{p\to\infty} \|x\|_p = \max_{1\le j\le d} |x_j| \ .$$

One defines

$$\|x\|_\infty := \max_{1\le j\le d} |x_j| \ .$$

b) Prove that for any finite interval $[a, b]$

$$L_p[a, b] \subset L_q[a, b]$$

provided that $q \le p$.

Problem 2: Let X be a normed space. Recall that a set $A \subset X$ is open, if for every $x_0 \in A$ there exists $r > 0$ so that the open ball $B_r(x) = \{x : \|x - x_0\| < r\} \subset A$. A set B is closed if its complement in X is open.

a) Show that a set B is closed if and only if for any convergent sequence $x_n \in B$, its limit is also in B.

b) Show that an arbitrary union of open sets is open and an arbitrary intersection of closed sets is closed.

Problem 3: Show that any two norms $\|\cdot\|_1, \|\cdot\|_2$ on $\mathbb{R}^d$ are **equivalent**, i.e., there exist constant c, C such that

$$c\|x\|_1 \le \|x\|_2 \le C\|x\|_1$$

for all $x \in \mathbb{R}^d$.

Problem 4: a) Fix $1 \le p < \infty$ and consider the space ℓ_p, the space of sequence $x = (x_1, x_2, \dots)$ such that

$$\sum_{j=1}^{\infty} |x_j|^p < \infty \ ,$$

with norm

$$\|x\|_p = \left(\sum_{j=1}^{\infty} |x_j|^p \right)^{1/p} .$$

Show that ℓ_p is complete.

b) Show also that ℓ_∞, which is the set of all bounded sequences with norm

$$\|x\|_\infty = \sup_{1 \le j < \infty} |x_j| ,$$

is complete.