SOME “PHILOSOPHICAL” REMARKS ABOUT LIMITS

I would like to make some simple remarks about the notion of limit and the real numbers. This is the point of view of a physicist or engineer. To be concrete consider the problem of measuring some rod, maybe made out of platinum. Our problem is to measure its length. I do not want to enter into a discussion about the physics of measuring length. Originally (1889) the meter was defined by a platinum-iridium rod. The modern definition of the meter (1983) is the distance it takes for light to travel in vacuum in 1/299792458 of a second.

The first conceptual problem is the existence of a length. Carpenters use their measuring devices and come up with an approximate length. Then, with more sophistication the precision can be improved until we reach the atomic scale. The surface of the platinum rod at this scale is quite rugged and hence one is tempted to argue that the rod has no length, which, from a practical point of view is ridiculous. In other words the notion of a length is related to the accuracy of our measurements.

Only measurements that improve the accuracy are of interest. The carpenter measures the length as $\ell_1 \pm \delta_1$. What this says is, that all subsequent measurements (which have to be more precise for otherwise they are not interesting) must fall in the interval $(\ell_1 - \delta_1, \ell_1 + \delta_1)$. With a more sophisticated device, the accuracy can be improved with the result $\ell_2 \in (\ell_1 - \delta_1, \ell_1 + \delta_1)$ for the length with an improved accuracy δ_2. Clearly, $\delta_2 < \delta_1$. Again, all subsequent measurements must fall in the interval $(\ell_2 - \delta_2, \ell_2 + \delta_2)$. In an ideal world $(\ell_2 - \delta_2, \ell_2 + \delta_2) \subset (\ell_1 - \delta_1, \ell_1 + \delta_1)$. Please note, that a measurement consists of two numbers, the “length” ℓ and the error δ. The length ℓ alone does not make any sense. With further improvements one can continue this process. As I said before, the process of measuring the length of a platinum rod must end at the atomic scale. Thus, one is tempted to assume that the ultimate error in length measurements is the size of an atom. This, of course, does not work. One can measure sizes of nuclei ($10^{-15}m$), in fact in the recent LIGO experiments the accuracy was driven to $10^{-19}m$. This accuracy will, presumably lead to a new definition of the meter.

The point of mathematics is to analyze interesting idealizations. A central one is the notion of limit, in which the sequence of measurements never terminates and the errors δ_n tend to zero with n. This means that for a given accuracy δ there exists a measurement ℓ_n so that all subsequent measurements fall into the interval $(\ell_n - \delta, \ell_n + \delta)$. This is precisely the notion of a **Cauchy sequence**. In its most elementary form it is realized by the decimal system. With the accuracy 0.1 one determines the digit before the decimal point. With the accuracy 0.01 one determines the first digit after the decimal point etc. In some sense we can say that analysis is engineering with infinite accuracy.

Such Cauchy sequence, or more properly their equivalence classes, lead to the definition of real numbers. Note, that such numbers are mostly a fiction but a useful one. To compute a real number means to give a procedure that yields for any δ the ℓ_n so that any approximation with hight accuracy is in the interval $(\ell_n - \delta, \ell_n + \delta)$. E.g., to compute $\sqrt{2}$ the Babylonian algorithm

$$\ell_{n+1} = \frac{1}{2}\left(\ell_n + \frac{2}{\ell_n}\right) \ , \ \ell_0 = 1$$

produces such a sequence. By procedure one assumes that the work can in principle be done by a Turing machine. It is important to notice that it can take only finitely many steps for each digit! The undefined word is 'steps' and that is where 'Turing machines' enter.

It is not hard to see that "most" of the real numbers are not computable in the sense just explained. In fact the computable numbers form a countable set whereas the real numbers are not countable. This seems to be a bit counterintuitive, because all one has to do is to write down a decimal expansion

$$N.p_1p_2p_3\cdots \ .$$

Thus, the set of real numbers is nothing but the collection of infinite decimal expansions. The problem with this description is that in general there is no rule that generates the decimal expansion. This is in contrast to the Babylonian algorithm that is a rule for computing the decimal expansion for $\sqrt{2}$. In fact, the notion of the set of real numbers is intimately connected with the axiom of choice. I do not want to get into this but it is good to keep in mind that most of the real numbers are not accessible. To invent ways of computing certain numbers is in the realm of numerical mathematics. Nevertheless, the notion of real numbers is a useful one and the notion that we can talk about a limit for every Cauchy sequence is called the **Completeness of the real numbers**. It is not hard to see that this notion is equivalent to the fact that every set of real numbers that is bounded above has a least upper bound and every set of real umbers that is bounded below has a greatest lower bound. We call these numbers supremum and infimum.

The completeness of the real numbers entails the notion of **existence**. For instance, consider the theorem that any function continuous on a closed interval attains its maximum and its minimum on that interval. In order to prove this let us be a bit more formal. We have a bounded closed interval $I = [a, b]$ and a function

$$f : I \to \mathbb{R}$$

and the function is continuous, i.e., for any sequence $\{x_n\}_{n=1}^{\infty}$, $x_n \in I$ such that $\lim_{n\to\infty} x_n = x$,

$$\lim_{n\to\infty} f(x_n) = f(x) \ .$$

Note, that since I is closed $x \in I$. The **existence** of a minimum can be proved as follows. Pick a minimizing sequence $x_n \in I$, i.e., a sequence such that

$$\lim_{n\to\infty} f(x_n) = \inf_I f \ .$$

Please note, that nobody tells you how to find that sequence. It follows from the notion of infimum that such a sequence exists. Now we use the fact that a bounded closed interval is compact, i.e., every sequence has a subsequence that converges. Denoting this sequence again by x_n we get that this sequence converges to some point y and

$$\inf_I f = \lim_{n \to \infty} f(x_n) = f(y)$$

which proves the claim. Again, from a computational point of view this theorem does not say much. We have no prescription whatsoever for how to find y. Intuitively, the word "existence" is tenuous from this perspective. The advantage, however, is that, in the realm of the real numbers, we can **talk** about the minimum and at a further stage for particular examples devices algorithms that allows us to compute them. To actually compute the minimum or maximum depends on the particular function and such problem can be very difficult.

Please note, that this scheme assumes a logical system. A constructivist who does not accept the real numbers as given, but only those that can be computed will not accept proofs using this scheme. In this course, however, we will not worry about these issues, but it is a good idea to keep these in mind. Often we say that certain mathematical objects exist but we have not the slightest idea how to compute them.