THE THEOREM OF ARZELÁ AND ASCOLI

As usual we endow $C[a,b]$ with the maximum norm. A subset $M \subset C[a,b]$ of functions is called **equicontinuous family** if for any $\varepsilon > 0$ there exists $\delta > 0$ sucht that for all x, y with $|x-y| < \delta$ and all $f \in M$ it follows that $|f(x) - f(y)| < \varepsilon$. Note that δ depends only on ε and not on $f \in M$ and not on x, as long as $|x - y| < \delta$. Often this is also called **uniform equicontinuity**. Recall that a set $M \subset C[a,b]$ is compact if any bounded sequence in M has a subsequence that converges to some f in $C[a,b]$.

Theorem 0.1 (Arzelá-Ascoli)**.** *A subset $M \subset C[a,b]$ is compact if and only if is a bounded and equicontinuous family.*

Proof. Enumerate the rational points in the interval $[a,b]$ and denote this set $\{r_1, r_2, r_3, \dots\}$ by Q. The sequence $f_n(r_1)$ is bounded and hence it has a convergent subsequence $f_n^{(1)}(r_1)$. The sequence $f_n^{(1)}(r_2)$ is again bounded and hence has a convergent subsequence $f_n^{(2)}(r_2)$. The sequence $f_n^{(2)}(r_3)$ is bounded and hence has a convergent subsequence $f_n^{(3)}(r_3)$ etc. Now consider the sequence $f_n^{(n)}(x)$. This is a sequence of functions that converges on all rational points of $[a,b]$ to some function $f(r_j)$. Note, so far we have only used that $f_n(x)$ is bounded. We may think of these limits as a function on the rational numbers in the interval $[a,b]$. We claim that the sequence $f_n^{(n)}(x)$ is a uniform Cauchy Sequence. To see this fix any $\varepsilon > 0$. There exists $\delta > 0$ so that $|f^{(n)}(x) - f^{(n)}(y)| < \varepsilon/3$ for all n and all x whenever $|x - y| < \delta$. This is precisely the assumption of equicontinuity. Choose rational points $\{p_1, \dots, p_M\}$ in such a way that they subdivide the interval $[a,b]$ in intervals of length less than δ. There exists N so that for all $n > N$,

$$|f_n^{(n)}(p_j) - f(p_j)| < \varepsilon/3$$

for all $j = 1, \dots, M$. It is worth remembering that N depends only on ε and not, e.g., on the choice of the rational points. Now we estimate

$$|f_n^{(n)}(x) - f_m^{(m)}(x)| \le |f_n^{(n)}(x) - f_m^{(m)}(p_j)| + |f_n^{(n)}(p_j) - f_m^{(m)}(p_j)| + |f_n^{(n)}(p_j) - f_m^{(m)}(x)|$$

where p_j is the the point closest to x. Hence

$$|f_n^{(n)}(x) - f_m^{(m)}(x)| < \varepsilon$$

for all $n > N$ and all x. Thus, the sequence $f_n^{(n)}(x)$ converges for all x to some limit which we denote by $f(x)$. Moreover the convergence is uniform and hence $f \in C[a,b]$. Further, we also know that for any $m > N$

$$|f(x) - f_m^{(m)}(x)| = \lim_{n\to\infty} |f_n^{(n)}(x) - f_m^{(m)}(x)| < \varepsilon$$

for all $x \in [a,b]$. Hence for any $m > N$

$$\|f - f_m^{(m)}\| \le \varepsilon$$

which proves the first part of the theorem.

To prove the converse, suppose the family M is not bounded. Then there exists a sequence of functions $f_n \in M$ and a sequence of points $x_n \in [a,b]$ such that $|f_n(x_n)|$ tends to infinity. Such a sequence, however, cannot have a subsequence that converges uniformly.

The equicontinuity is a bit more tricky. Suppose that M is not equicontinuous. What does this mean. Recall that M is equicontinuous if

"For any $\varepsilon > 0$ there exists $\delta > 0$ so that for all x, y with $|x - y| < \delta$ and all $f \in M$ we have that $|f(x) - f(y)| < \varepsilon$"

Suppose that M is not equicontinuous. The negation of the above statement says:

There exists some $\varepsilon_0 > 0$ such that for any $\delta > 0$ there exist x, y with $|x - y| < \delta$ and $f \in M$ with $|f(x) - f(y)| > \varepsilon_0$.

Hence, for this ε_0 we have sequences $\delta_n \to 0$, x_n, y_n and $f_n \in M$ such that

$$|f_n(x_n) - f_n(y_n)| > \varepsilon_0 \text{ whenever } |x_n - y_n| < \delta_n \ .$$

By assumption, this sequence f_n must have a convergent subsequence, which we denote again by f_n. This means that there exists $f \in C[a, b]$ such that

$$\max_{a \le x \le b} |f(x) - f_n(x)| \to 0 \ , n \to \infty \ .$$

In particular, the function f must be uniformly continuous, i.e., for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sup_{\{x,y: |x-y|<\delta\}} |f(x) - f(y)| < \varepsilon \ .$$

However, we have that $|x_n - y_n| \to 0$ as $n \to \infty$ and

$$|f(x_n) - f(y_n)| \ge |f_n(x_n) - f_n(y_n)| - |f(x_n) - f_n(x_n)| - |f(y_n) - f_n(y_n)| > \varepsilon_0/2$$

for n sufficiently large, which is a contradiction. $\square$