THE BANACH LIMIT

We have constructed all bounded linear functionals on c_0. Now one might expect, naively, that, since $c_0 \subset \ell_\infty$ and therefore any bounded linear functional on ℓ_∞ is also a bounded linear functional on c_0, we must have that $\ell_\infty^* \subset c_0^*$. This is not correct, as we shall see shortly. Let's go through the arguments and see where it might fail.,

Consider a bounded linear functional f on ℓ_∞. As before, define $b_j = f(e_j)$. We shall prove that

$$\sum_{=1}^{\infty} |b_j| < \infty ,$$

i.e., the sequence $(b_j) \in \ell_1$. To see this, consider

$$y_n = \sum_{j=1}^{n} \frac{b_j}{|b_j|} e_j \in \ell_\infty$$

and note that, as before, $\|y_n\|_{\ell_\infty} = 1$ and

$$f(y_n) = \sum_{j=1}^{n} |b_j| .$$

Recall that e_j is the sequence consisting of 1 in the j-th entry and zero otherwise. Hence

$$\|f\|_{\ell_\infty{}^*} \geq f(y_n) = \sum_{j=1}^{n} |b_j|$$

which shows that

$$\sum_{j=1}^{\infty} |b_j| \leq \|f\|_{\ell_\infty{}^*} .$$

Thus, we are tempted to say that $\ell_\infty^* = \ell_1$. This is, however, not correct. The functional f is in general *not* given by

$$f(x) = \sum_{i=1}^{\infty} b_i a_i .$$

Note, that the argument we used to establish this formula for c_0^* breaks down. Indeed, there are non-trivial linear functionals $f \in \ell_\infty$ that vanish on all of c_0!

The standard example is the **Banach limit** For $x = (a_i) \in \ell_\infty$ consider the linear functional

$$f_N(x) = \frac{\sum_{j=1}^{N} a_j}{N} .$$

Consider the subspace $E \subset \ell_\infty$ consisting of all sequences (a_j) such that the limit

$$\lim_{N \to \infty} \frac{\sum_{j=1}^{N} a_j}{N}$$

exists. On E define

$$f(x) = \lim_{N \to \infty} f_N(x)$$

which is a linear functional. Further we have that $|f(x)| \leq \|x\|_{\ell_\infty}$ and if $x_0 = (1, 1, \dots)$, then $f(x_0) = 1 = \|x_0\|_{\ell_\infty}$. Thus, f is a linear functional on E and $\|f\|_E = 1$. By H.B. there exists f_B a linear functional on ℓ_∞ such that $f_B = f$ on E and f_B and f have the same norm. In particular f_B is not the zero functional. Note, however, that $f(x) = 0$ for all $x \in c_0$ and hence $f_B(x) = 0$ on c_0. Another interesting fact is that on can find an extension f_B has the property that it is invariant against shifts. Let $T(x) = (x_2, x_3, \dots)$. Then

$$f_B(T(x)) = f_B(x) \ .$$

This is a bit trickier to see.

Two points to be made: The extensions whose existence the H.B. delivers are in general not unique. The naive idea that if $E \subset X$ then $X^* \subset E^*$ is in general wrong