BANACH SPACES, I.E., COMPLETE NORMED SPACES

In most applications one has an additional structure in that the underlying space is a vector space. We denote by $\mathbb{R}_+$ the nonnegative real numbers. The vector space may be over the reals or the complex numbers. A **norm** on a vector space N is a function

$$\|\cdot\| : N \to \mathbb{R}_+$$

having the following properties:

$$\|x\| = 0 \Leftrightarrow x = 0$$

$$\|\lambda x\| = |\lambda| \|x\|$$

$$\|x + y\| \le \|x\| + \|y\| \text{ all } x, y \in N$$

A verctor space N with a norm is called a **normed vector space**.

Example: For any set $S \subset \mathbb{R}^d$ consider the continuous functions $f : S \to \mathbb{R}$. The set of such functions is denoted by $C(S)$. On this set consider the norm

$$\|f\| = \sup_{x \in S} |f(x)|$$

which leads to the distance

$$d(f, g) = \|f(x) - g(x)\| \ .$$

If S is compact, the sup is a max but that is not relevant for the moment. It is easy to see that d defined in this way turns the set $C(S)$ into a metric space, in fact a Banach space. We shall show that this space is complete.

Proof. We show first that any Cauchy sequence $f_n \in C(S)$ converges to a continuous function. To see this, note that for any $x \in S$, the sequence $f_n(x)$ is a Cauchy sequence of real numbers and hence converges to a number which we call $f(x)$. In this way we obtain a function on S that is the point-wise limit of the sequence f_n. This function f is continuous. Indeed, pick any $\varepsilon > 0$. We have to find $\delta > 0$ such that for all $x, y \in S$ with $|x - y| < \delta$ it follows that $|f(x) - f(y)| < \varepsilon$. There exists $N(\varepsilon)$ so that for all $x \in S$ and $n > N(\varepsilon)$

$$|f(x) - f_n(x)| < \varepsilon/3 \ .$$

Indeed, pick $N(\varepsilon)$ such that

$$\sup_{x \in S} |f_n(x) - f_m(x)| < \varepsilon/3$$

for all $n, m > N(\varepsilon)$ and note that for any $x \in S$

$$|f(x) - f_n(x)| = \lim_{m \to \infty} |f_m(x) - f_n(x)| \le \varepsilon/3 \ .$$

In particular

$$\sup_{x \in S} |f(x) - f_n(x)| < \varepsilon/3 \tag{1}$$

whenever $n > N(\varepsilon)$. Fix x and pick such an n and note that the exists $\delta > 0$ so that for all $y \in S$ with $|x - y| < \delta$ we have that

$$|f_n(x) - f_n(y)| < \varepsilon/3 \ .$$

This follows, because f_n is continuous. Hence for all such y

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| < \varepsilon \ .$$

This shows that f is continuous. The fact that

$$\lim_{n \to \infty} d(f_n, f) = 0 \ .$$

follows from (1). That $\|f\| < \infty$ follows from

$$|f(x)| \leq |f(x) - f_n(x)| + |f_n(x)|$$

so that

$$\|f\| \leq \|f - f_n\| + \|f_n\| < \infty \ .$$

□

1. Existence and uniqueness of solutions for differential equations

We consider a differential equation of the form

$$\dot{x} = v(x)$$

where $v : \mathbb{R}^d \to \mathbb{R}^d$. A function $\phi : I \to \mathbb{R}^d$ solves an initial value problem if it satisfies the above equation and in addition the conditions $\phi(0) = x_0$. (Likewise we also can consider the initial value problem with $\phi(t_0) = x_0$.) The interval I contains the initial time $t = 0$. We shall assume that the vector filed v is Lipschitz, i.e., there exists a constant L so that for all $x, y \in \mathbb{R}^d$

$$|v(x) - v(y)| \leq L|x - y| \ .$$

We shall show that this problem has a unique solution for all times. The uniqueness part is easy and is left as an exercise. The existence is more interesting. We convert the problem into an integral equation.

$$\phi(t) = x_0 + \int_0^t v(\phi(s)) ds$$

and apply the Banach fixed point theorem. We shall consider the set of all continuous function $\psi : I \to \mathbb{R}^d$, except we do not yet know how big the interval I should be. We require that these functions satisfie $\psi(0) = x_0$ and we endow this space with the norm

$$\sup_{t \in I} |\psi_1(t) - \psi_2(t)| \ .$$

We have seen before that this space is a complete metric space. First we do a few obvious computations. We define

$$F(\phi)(t) := x_0 + \int_0^t v(\phi(s)) ds$$

and find that

$$|F(\phi)(t) - x_0| \leq \int_0^t |v(\phi(s)) - v(x_0)| ds + t|v(x_0)| \leq L \int_0^t |\phi(s) - x_0| ds + t|v(x_0)|$$

so that

$$|F(\phi)(t) - x_0| \leq t \left[L \sup_{0 \leq s \leq t} |\phi(s) - x_0| + |v(x_0)| \right] \ .$$

For a given number $R > 0$ choose $t_0 > 0$ so that

$$t_0[LR + |v(x_0)|] < R$$

i.e.,

$$t_0 < \frac{R}{LR + |v(x_0)|} \ .$$

Thus, if we fix $R > 0$ then the closed ball

$$\overline{B}_R(x_0) = \{\phi : [-t_0, t_0] \to \mathbb{R}^d : \sup_{-t_0 \le t \le t_0} |\phi(t) - x_0| \le R\}$$

is mapped into itself by F. Next, we compute

$$|F(\psi_1)(t) - F(\psi_2)(t)| \le tL \sup_{0 \le s \le t} |\psi_1(s) - \psi_2(s)|$$

which once more implies that

$$\sup_{-t_0 \le t \le t_0} |F(\psi_1)(t) - F(\psi_2)(t)| \le t_0 L \sup_{-t_0 \le t \le t_0} |\psi_1(t) - \psi_2(t)|$$

and we see that the map F is a contraction on $\overline{B}_R(x_0)$ provided we choose t_0 such that $Lt_0 < 1$. By the Banach fixed point theorem there exists a unique function $\phi(t)$ continuous on the interval $[-t_0, t_0]$ such that $F(\phi)(t) = \phi(t)$, in particular

$$\phi(t) = x_0 + \int_0^t v(\phi(s)ds \ .$$

Since ϕ is continuous and v Lipschitz, ϕ is differentiable and solves the differential equation on the interval. Since the Lipschitz condition is uniform one can continue the solution indefinitely and obtains a global solution.