COMPACT OPERATORS ON A SEPARABLE HILBERT SPACE

Compact operators on a separable Hilbert space have an additional property not shared by compact operators on a Banach space. On a separable Hilbert space, they can be viewed as norm limits of finite rank operators. In other words he smallest closed space (closed in the operator norm) of linear operators that contains the finite rank operators are the compact operators. In applications, most of the Hilbert spaces are separable and hence much of the analysis can be reduced to operators with finite rank. This is important for applications, since computations for compact operators can be reduced to computations with operators of finite rank. To prove this, we start with a simple lemma.

Lemma 0.1. *Let H be a separable Hilbert space and let $K : H \to H$ be a compact operator. Further, let $\{e_n\}_{n=1}^\infty$ be an orthonormal basis and define*

$$\mu_N = \sup\{\|Kx\| : x \perp \ \text{span}[e_1, \dots, e_N] \ , \|x\| = 1\} \ .$$

Then $\lim_{N\to\infty} \mu_N = 0$.

Proof. The sequence μ_N is decreasing and hence has a limit μ. Suppose that $\mu > 0$. For each N there exists a vector x_N with $\|x_N\| = 1$, such that x_N is perpendicular to the span of the vectors $e_1, \dots, e_N$ and such that

$$\|Kx_N\| \geq \mu/2 \ .$$

The sequence x_N is bounded and hence there exists a subsequence $x_N^{(1)}$ such that $Kx_N^{(1)}$ converges strongly to some vector y, i.e.,

$$\|Kx_N^{(1)} - y\| \to 0$$

as $N \to \infty$. Further

$$\begin{aligned}\|Kx_N^{(1)} - y\|^2 &= \|Kx_N^{(1)}\|^2 + \|y\|^2 - 2\langle Kx_N^{(1)}, y\rangle \\ &= \|Kx_N^{(1)}\|^2 + \|y\|^2 - 2\langle x_N^{(1)}, K^*y\rangle \ .\end{aligned}$$

Here K^* is the adjoint operator which is bounded. In fact it is compact but this is not relevant here. Since

$$|\langle x_N^{(1)}, K^*y\rangle| \leq \|x_N^{(1)}\| \|P_N K^*y\|$$

where P_N is the projection onto the subspace of all vectors perpendicular to $e_1, \dots, e_N$ and $\{e_n\}_{n=1}^\infty$ is a basis, we have that $\|P_N K^*y\| \to 0$ as $N \to \infty$ and hence

$$\lim_{N\to\infty} |\langle x_N^{(1)}, K^*y\rangle| = 0 \ .$$

More precisely,

$$\|P_N K^*y\|^2 = \sum_{n=N+1}^\infty |\langle e_n, K^*y\rangle|^2 \to 0$$

as $N \to \infty$. Thus, $\|y\| = 0$ and therefore

$$\lim_{N\to\infty} \|Kx_N^{(1)}\|^2 = 0$$

which is a contradiction. Hence $\mu = 0$ and the lemma is proved. □

Theorem 0.2. *Let H be a separable Hilbert space, and $K : H \to H$ a compact operator. For any $\varepsilon > 0$ there exists a finite rank operator K_ε such that*

$$\|K - K_\varepsilon\| < \varepsilon \ .$$

Proof. Fix a basis $\{e_n\}_{n=1}^{\infty}$. Such a basis exists, because H is a separable Hilbert space. Pick N such that

$$\mu_N = \sup\{\|Kx\| : x \perp \ \text{span}[e_1, \ldots, e_N] \ , \|x\| = 1\} < \varepsilon \ .$$

Such N exists on account of the previous lemma. Define

$$K_\varepsilon x = \sum_{n=1}^{N} (Ke_n)\langle x, e_n\rangle = K \sum_{n=1}^{N} e_n \langle x, e_n\rangle \ .$$

Then

$$\|(K - K_\varepsilon)x\| = \|KP_N x\|$$

where P_N is the projection onto the orthogonal complement of the space spanned by $e_1, \ldots, e_N$. But

$$\|KP_N x\| \le \mu_N \|x\| < \varepsilon\|x\|$$

and hence $\|K - K_\varepsilon\| < \varepsilon$. □