THE MIN-MAX AND MAX-MIN PRINCIPLE

Eigenvalues of linear operators are notoriously hard to compute and a considerable amount of research goes into estimating eigenvalue and coming up with numerical schemes for computing them in an efficient manner. The simplest way for estimating eigenvalues are the min-max and the max-min principles we now describe.

The setting is a Hilbert space H and a linear, compact and self-adjoint operator $A : H \to H$. The spectrum of A consists of $\{0\} \cup \sigma_p(A)$ and the eigenvalues, counted with their multiplicity are real and can only accumulate at 0. We denote the positive eigenvalues by $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ and the negative eigenvalues by $\mu_1 \leq \mu_2 \leq \cdots \leq 0$.

Recall that

$$\lambda_1 = \max\{\langle Ax, x\rangle : x \in H, \|x\| = 1\} \ ,$$

and any $x_1, \|x_1\| = 1$ with $\langle Ax_1, x_1\rangle = \lambda_1$ satisfies $Ax_1 = \lambda_1 x_1$. Likewise,

$$\mu_1 = \min\{\langle Ax, x\rangle : x \in H, \|x\| = 1\} \ .$$

Similar expressions can be found for higher eigenvalues. The following theorems are stated only for the positive eigenvalues and we leave it to the reader to formulate them for the negative eigenvalues.

Theorem 0.1. *Min-max priniple Define the numbers*

$$\nu_n = \min\{\sup\{\langle Ax, x\rangle : x \perp M_n \ , \|x\| = 1\} : M_n \subset H, \mathrm{dim} M_n = n - 1\} \ .$$

Then $\nu_n = \lambda_n$.

Proof. Pick any subspace $M_n \subset H$ with $\mathrm{dim} M_n = n - 1$. Pick any vector

$$x = \sum_{j=1}^{n} c_j x_j$$

where $Ax_j = \lambda_j x_j$. we want to choose the numbers c_j such that $x \neq 0$ and $x \perp M_n$. Pick any basis $\{e_, \dots, e_{n-1}$ in M_n and consider the system of equations

$$\sum_{j=1}^{n} c_j \langle x_j, e_i\rangle = 0 \ , i = 1, \dots, n-1 \ .$$

These are $n - 1$ equations with n unknowns and hence there exists a nontrivial solution $d_1, \dots, d_n$. The vector

$$x = \sum_{j=1}^{n} d_j x_j$$

is perpendicular to M_n and nonzero and hence we may assume that it is normalized. Now

$$\langle Ax, x\rangle = \sum_{j=1}^{n} \lambda_j |c_j|^2$$

because the eigenvectors are ortho-normal. Recall that the λ_j are ordered in a decreasing fashion and hence

$$\langle Ax, x\rangle \geq \lambda_n \sum_{j=1}^{n} |c_j|^2 = \lambda_n \|x\|^2 \ .$$

Hence, we have shown that for any M_n,

$$\sup\{\langle Ax, x\rangle : x \perp M_n \ , \|x\| = 1\} \geq \lambda_n$$

and hence $\nu_n \geq \lambda_n$. The converse inequality follows by choosing $M_n = \mathrm{span}[x_1, \dots, x_{n-1}]$. Then

$$\sup\{\langle Ax, x\rangle : x \perp M_n \ , \|x\| = 1\} = \lambda_n$$

which implies that $\nu_n \leq \lambda_n$. □

Now we come to the max-min principle. This is in many ways more natural than the previous one.

Theorem 0.2. *Consider the number*

$$\tau_n = \sup\{\min\{\langle Ax, x\rangle : x \in N_n\} : N_n \subset H, \mathrm{dim} N_n = n\}$$

Then $\tau_n = \lambda_n$.

Proof. Let N_n be an arbitrary n-dimensional subspace of H and consider a basis $\{e_1, \dots, e_n\}$ of this space. Any vector $x \in N_n$ can be written as $x = \sum_{j=1}^{n} c_j e_j$. In fact, because we have n free coefficients we may choose them in such a way that x is normalized and perpendicular to $x_1, \dots, x_{n-1}$. Hence,

$$\langle Ax, x\rangle \leq \lambda_n$$

for every subspace N_n and hence $\tau_n \leq \lambda_n$. To obtain the reverse inequality we choose the space N_n spanned by the vectors $x_1, \dots, x_n$. Then $\min\{\langle Ax, x\rangle : x \in N_n\} = \lambda_n$ and hence $\tau_n \geq \lambda_n$. □

In applications one proceeds often in the following way. Choose any n-dimensional subspace N_n and fix an orthonormal basis $\{e_1, \dots, e_n\}$ in N_n. Now form the matrix B consisting of the matrix elements $\langle Ae_i, e_j\rangle$. This $n \times n$ matrix is self-adjoint and hence can be diagonalized with eigenvalues $r_1 \geq r_2 \geq \cdots \geq r_n$. Theorem can be applied to show that

$$\lambda_1 \geq r_1 \ , \ \lambda_2 \geq r_2 \ , \ \cdots \ , \lambda_n \ \geq r_n \ .$$

Diagonalizing B yields eigenspaces $E_1, \dots, E_n$. Applying Theorem with $N_1 = E_1$ yields the first inequality. Then we choose $N_2 = E_1 \oplus E_2$ which yields the second inequality and so on. While this procedure delivers lower bounds, we do not know anything about how good these bounds are. There is no general method to get upper bounds on the first eigenvalue of A.

Recall that the set of self adjoint operators is partially ordered. We say that $A \geq B$ if for any $x \in H$ we have that

$$\langle Ax, x\rangle \geq \langle Bx, x\rangle \ .$$

The min-max or max-min theorems about immediately imply the

Theorem 0.3. *Let A, B be compact self adjoint operator and assume that $A \geq B$. Denote the eigenvalues of A by $\lambda_1 \geq \lambda_2 \geq \cdots$ and the eigenvalues of B by $\mu_{\geq} \mu_2 \geq \cdots$. Then*

$$\lambda_1 \geq \mu_1 \ , \lambda_2 \geq \mu_2 \ , \cdots \ .$$

Proof. We apply Theorem to the spaces $N_n = E_n$ where E_n is the space spanned by the first n eigenvectors of B. It follows that

$$\lambda_n \geq \min\{\langle Ax, x\rangle : x \in E_n\} \geq \min\{\langle Bx, x\rangle : x \in E_n\} = \mu_n \ .$$

$\square$