PROOF OF THE HAHN-BANACH THEOREM

We first start with the real case. A convenient notion is the one of a **sublinear function** p. This is a non-negative function that satisfies

$$p(\lambda x) = \lambda p(x) \ , \lambda \geq 0$$

and

$$p(x+y) \leq p(x) + p(y) \ .$$

Note that we do not require that $p(x) = p(-x)$, i.e., that p is symmetric. An example of a sublinear function is a norm. This, however, is a much stronger notion, since we have that $\|\lambda x\|| = \lambda \|x\|$ also for $\lambda \leq 0$.

Theorem 0.1. *Let L be a linear space over the real numbers and $p(x)$ be a sublinear function on L. Let L_0 be a linear subspace of L and f_0 be a linear functional defined on L_0. Assume that*

$$f_0(x) \leq p(x) \text{ for all } x \in L_0 \ .$$

Then there exists a linear functional f defined on L such that

$$f(x) = f_0(x) \text{ for all } x \in L_0$$

and

$$f(x) \leq p(x) \text{ for all } x \in L \ .$$

Note that if $f(x) \leq p(x)$ and $-f(x) = f(-x) \leq p(-x)$. Hence, such a functional is in some sense bounded.

Proof. We start by going from a the space L_0 up by one dimension. Pick any vector $y \notin L_0$ and consider the span $L_1 = \langle y, L \rangle$. If $z \in L_1$ we can write it in a unique way as

$$z = \lambda y + x \ , x \in L_0 \ .$$

Hence,

$$f_1(z) = \lambda f_1(y) + f_1(x) = \lambda f_1(y) + f_0(x) \ .$$

Our problem is, to choose $C := f_1(y)$ in such a way that $f_1(z) \leq p(z)$ for all $z \in L_1$, i.e.,

$$\lambda C + f_0(x_1) \leq p(\lambda y + x_1) \ , \lambda \geq 0$$

as well as

$$-\lambda C + f_0(x_2) \leq p(-\lambda y + x_2) \ , \lambda \geq 0 \ .$$

Let us write out these inequalities in the form

$$-p(-y + \frac{x_2}{\lambda}) + \frac{f_0(x_2)}{\lambda} \leq C \leq p(y + \frac{x_1}{\lambda}) - \frac{f_0(x_1)}{\lambda} \ .$$

Here we have assumed that $\lambda > 0$, the case $\lambda = 0$ being trivial. Hence, our problem is to show that for all $\lambda > 0$ and all $x_1, x_2 \in L_0$ we have that

$$-p(-y + \frac{x_2}{\lambda}) + \frac{f_0(x_2)}{\lambda} \leq p(y + \frac{x_1}{\lambda}) - \frac{f_0(x_1)}{\lambda}$$

or by setting $u_j = x_j/\lambda \in L_0$ we have that show that for all $u_1, u_2 \in L_0$ we have that

$$-p(-y+u_2) + f_0(u_2) \le p(y+u_1) - f_0(u_1) \ .$$

Once this is established, we have that

$$\sup_{u \in L_0} -p(-y+u) + f_0(u) \le \inf_{u \in L_0} -p(-y+u) + f_0(u)$$

we may choose the C between these values. In other words we have to show that

$$f_0(u_1+u_2) \le p(y+u_1) + p(-y+u_2) \ .$$

This follows from

$$f_0(u_1+u_2) \le p(u_1+u_2) = p(y+u_1-y+u_2) \le p(y+u_1) + p(-y+u_2) \ .$$

The first inequality holds because $u_1+u_2 \in L_0$ and the second follows from the subadditivity. Thus, we have found a pair (L_1, f_1) with the desired properties. Now, consider the collection of all pairs $P := \{(L_alpha, f_\alpha)\}$ such that $f_\alpha(x) \le p(x), x \in L_\alpha$ and $f_\alpha(x) = f_0(x), x \in L_0$. This set is not empty. On this set P we introduce a partial ordering by setting

$$(L_\alpha, f_\alpha) \prec (L_\beta, f_\beta)$$

if L_α is a subspace of L_β and $f_\beta(x) = f_\alpha(x), x \in L_\alpha$. For any linearly ordered chain there is a supremum element $(L_\infty, f_\infty) \in P$ given by

$$L_\infty = \cup_{\alpha \in chain} L_\alpha$$

and if $x \in L_\infty$ then $x \in L_\alpha$ for some $\alpha \in chain$ so that we can set

$$f_\infty(x) = f_\alpha(x) \ .$$

By Zorn's lemma there exists a maximal element $(L_{\max}, f_{\max})$. We have to show that $L_{\max} = L$. Suppose not. Then there exists $y \in L$ but $y \notin L_{\max}$. define the subspace L_1 as before and construct as before the functional f_1. As before we can verify all the properties of the pair (L_1, f_1) and conclude that $(L_{\max}, f_{\max}) \prec (L_1, f_1)$ which contradicts the maximality of $(L_{\max}, f_{\max})$. hence $L_{\max} = L$ and $f_{\max}$ is the desired extension. □