SEMI-CONTINUITY

In this section we would like to collect some facts about semi continuous functions. We keep the exposition informal and leave some of the straightforward proofs to the reader.

Recall that, quite generally, a real function f from some topological space S into the reals is continuous if for any open subset $\mathcal{O}$ of the reals, the inverse image, i.e,

$$f^{-1}(\mathcal{O}) = \{x \in S : f(x) \in \mathcal{O}\}$$

is open in S. It suffices to restrict $\mathcal{O}$ to an open interval, because every open subset of the reals is a countable union of open intervals. Thus, if $\mathcal{O} = \cup_{j=1}^{\infty} I_j$ then

$$\{x \in S : f(x) \in \mathcal{O}\} = \cup_{j=1}^{\infty}\{x \in S : f(x) \in I_j\}$$

The fact that $f^{-1}(I)$ is open for any interval is equivalent to both, $f^{-1}((t,\infty))$ and $f^{-1}((-\infty,t))$ being open for all $t \in \mathbb{R}$.

Suppose that $f_n : S \to \mathbb{R}$ is a sequence of continuous functions with $f_n(x) \geq f_{n+1}(x)$. What can be said about the limit $f(x) = \lim_{n\to\infty} f_n(x)$ provided it exists? Note that the sets

$$U_n = \{x \in S : f_n(x) > t\}$$

are open sets and $U_{n+1} \subset U_n$, whereas

$$V_n = \{x \in S : f_n(x) < t\}$$

satisfy $V_n \subset V_{n+1}$. Hence we conclude that

$$\{x \in S : f(x) < t\} = \cup_{n=1}^{\infty} V_n$$

is open. However, nothing can be said about

$$\{x \in S : f(x) > t\} = \cap_{n=1}^{\infty} U_n \ .$$

A function with the property that $f^{-1}((-\infty,t))$ is open for all $t \in \mathbb{R}$ is an **upper semi-continuous function**. Likewise, a function for which $f^{-1}((t,\infty))$ is open is a **lower semi-continuous function**. We see that upper semi-continuous functions are preserved under monotone decreasing limits and lower semi-continuous functions are preserved under monotone increasing limits. Indeed, suppose that $\{f_\alpha(x)\}_{\alpha\in I}$ is a family of upper semi-continuous functions. Then

$$\{x \in S : \inf_{\alpha\in I} f_\alpha(x) \geq t\} = \cap_{\alpha\in I}\{x \in S : f_\alpha(x) \geq t\}$$

and since $\{x \in S : f_\alpha(x) \geq t\}$ is closed so is $\{x \in S : \inf_{\alpha\in I} f_\alpha(x) \geq t\}$. The proof for lower semi-continuous functions is similar. If the topological space S is a metric space then a function $f : S \to \mathbb{R}$ is lower semi-continuous if and only if for any $x \in S$ and any sequence x_n converging to z,

$$\liminf_{x_n\to z} f(x_n) \geq f(z) \ . \tag{1}$$

Denote the left side of the above inequality by t and suppose that f is l.s.c. and the above statement is false, i.e. $t < f(z)$ Then there exists $\varepsilon > 0$ such that $t + \varepsilon < f(z)$. There exists a subsequence (again denoted by x_n) so that $\lim_{x_n\to z} f(x_n) = t$ and hence all but finitely many elements of this sequence are in the set

$$C := \{x \in S : f(x) \leq t + \varepsilon/\}$$

Since this set is closed, $z \in C$ and hence $f(z) \leq t + \varepsilon/2$ which is a contradiction. Conversely, assume that for every $z \in S$ and any sequence x_n converging to z we have that

$$t := \liminf_{x_n \to z} f(x_n) \geq f(z) \ .$$

Pick any sequence x_n in the set $C := f^{-1}((-\infty, t])$ such that $x_n \to z$. We have to show that $z \in C$. There exists a subsequence (again denoted by x_n) such that $f(x_n) \to t$. Hence all but finitely many elements of the sequence are in $f^{-1}((-\infty, t + 1/m])$ and hence $z \in f^{-1}((-\infty, t + 1/m])$ since $f(z) \leq t$. Hence

$$z \in \cap_{m=1}^{\infty} f^{-1}((-\infty, t + 1/m]) = f^{-1}((-\infty, t]) \ ,$$

and hence the set $f^{-1}((-\infty, t])$ is closed. The condition analogous the (1) for upper semicontinuous functions is

$$\limsup_{x_n \to x} f(x_n) \leq f(x)$$

and the reasoning for the proof is similar to one for lower semicontinuous functions.

Another useful property is that semicontinuous functions on compact sets attain some of their extrema. More precisely, an upper semicontinuous function defined on a compact set attains its supremum on this set wheras a lower semicontinuous functions attains its infimum on the set. The proofs use (1) and are very easy. Another interesting fact about semi continuous functions is that they can be approximated by continuous functions. In fact we have:

Lemma 0.1. *Assume that S is a compact metric space and let $f : S \to \mathbb{R}$ be an upper semicontinuous function. Then the function*

$$f_\varepsilon(x) = \sup_{y \in S} \left(f(y) - \frac{d(x, y)}{\varepsilon} \right)$$

is continuous and converges pointwise to $f(x)$ from above as ε tends to zero. Moreover, if $\varepsilon le \varepsilon'$, then

$$f_\varepsilon(x) \leq f_{\varepsilon'}(x) \ .$$

Proof. Clearly, $f(x) \leq f_\varepsilon(x)$ by setting $y = x$ instead of taking the supremum. For each fixed $y \in S$ the function

$$x \to f(y) - \frac{d(x, y)}{\varepsilon}$$

is continuous and hence $f_\varepsilon(x)$, i.e., the supremum, is a lower semicontinuous function. Because S is compact, the function f is upper semicontinuous and $y \to d(x, y)$ is continuous, the supremum is attained and there exists $y(\varepsilon, x) \in S$ such that

$$f_\varepsilon(x) = f(y(\varepsilon, x)) - \frac{d(x, y(\varepsilon, x))}{\varepsilon} \ .$$

Pick any sequence $x_n \to x$. Since S is compact we may assume, by passing to a subsequence, that $y(\varepsilon, x_n) \to y_0$. Then, since $f_\varepsilon(x)$ is lower semicontinuous,

$$f_\varepsilon(x) \leq \liminf_{n \to \infty} f_\varepsilon(x_n) = \liminf_{n \to \infty} \left(f(y(\varepsilon, x_n)) - \frac{d(x_n, y(\varepsilon, x_n))}{\varepsilon} \right)$$

$$= \liminf_{n \to \infty} f(y(\varepsilon, x_n)) - \frac{d(x, y_0)}{\varepsilon} \leq f(y_0) - \frac{d(x, y_0)}{\varepsilon} \leq f_\varepsilon(x) \ .$$

Hence $f_\varepsilon(x)$ is a continuous function. Moreover, as $\varepsilon \to 0$, $y(\varepsilon, x) \to x$, otherwise the second term would tend to $-\infty$. Moreover,

$$\frac{d(x, y(\varepsilon, x))}{\varepsilon} \to 0$$

as $\varepsilon \to 0$. This follows from the inequalities

$$f(x) \le \liminf_{\varepsilon \to 0} f_\varepsilon(x) = \liminf_{\varepsilon \to 0} \left(f(y(\varepsilon, x)) - \frac{d(x, y(\varepsilon, x))}{\varepsilon} \right) \le f(x) - \limsup_{\varepsilon \to 0} \frac{d(x, y(\varepsilon, x))}{\varepsilon}$$

which also proves that $f_\varepsilon(x)$ converges pointwise to $f(x)$. Finally, for $\varepsilon \le \varepsilon'$,

$$f_\varepsilon(x) = \sup_{y \in S} \left(f(y) - \frac{d(x,y)}{\varepsilon'} + \frac{d(x,y)}{\varepsilon'} - \frac{d(x,y)}{\varepsilon} \right) \le \sup_{y \in S} \left(f(y) - \frac{d(x,y)}{\varepsilon'} \right) = f_{\varepsilon'}(x) \ .$$

□

Needless to say that for a lower semicontinuous function $f : S \to \mathbb{R}$ where S is again a compact metric space, we use a continuous approximation

$$f_\varepsilon(x) = \inf_{y \in S} \left(f(y) + \frac{d(x,y)}{\varepsilon} \right) \ .$$

Another, sometimes useful fact is the following lemma.

Lemma 0.2. *Let S be a compact metric space and $f : S \to \mathbb{R}$ a continuous function. Let $f_n : S \to \mathbb{R}$ be a sequence of continuous functions that converge monotonically to f. Then the convergence is uniform.*

Proof. We may assume that $f_{n+1}(x) \le f_n(x)$, i.e., the sequence is decreasing. Fix any $\varepsilon > 0$ and pick $x \in S$ arbitrary. There exists $N(x)$ such that $|f(x) - f_n(x)| < \varepsilon$ for all $n > N(x)$. More precisely

$$f_n(x) - \varepsilon < f(x) \le f_n(x) \ .$$

Set $n = N(x) + 1$. There exists an open ball centered at x with radius $r(x)$ such that this inequality holds for all x in this ball $B(x, r(x))$. Because the convergence is monotone we find that for any $m \ge N(x) + 1$

$$f_m(x) - \varepsilon < f(x) \le f_n(x)$$

for all $x \in B(x, r(x))$. Since $x \in S$ is arbitrary we have that

$$S = \cup_{x \in S} B(x, r(x))$$

and since S is compact there exists a finite sub-cover,

$$S = B(x_1, r(x_1)) \cup B(x_2, r(x_2)) \cdots \cup B(x_M, r(x_M))$$

Now pick $N = \max\{N(x_1), \dots, N(x_M)\}$ and for any $n > N$ we have that

$$f_n(x) - \varepsilon < f(x) \le f_n(x)$$

and the convergence is uniform.

□