THE WEIERSTRASS APPROXIMATION THEOREM

There is a lovely proof of the Weierstrass approximation theorem by S. Bernstein. We shall show that any function, continuous on the closed interval $[0,1]$ can be uniformly approximated by polynomials. We start with the building blocks, the Bernstein polynomials which are given by the expressions

$$B_{n,k}(x) = \left(\begin{array}{c} n \\ k \end{array}\right) x^k(1-x)^{n-k} \ , k = 0, 1, \dots, n \ .$$

As always

$$\left(\begin{array}{c} n \\ k \end{array}\right) = \frac{n!}{k!(n-k)!} \ .$$

Note that $B_{n,k}(x) \geq 0$ on $[0,1]$. E.g.,

$$B_{1,0} = (1-x) \ , \ B_{1,1} = x \ ,$$

$$B_{2,0} = (1-x)^2 \ , \ B_{2,1} = 2)x(1-x) \ , \ B_{2,2} = x^2 \ .$$

Here are some simple identities.

Lemma 0.1. *We have the following formulas*

$$\sum_{k=0}^{n} B_{n,k}(x) = 1 \ ,$$

$$\sum_{k=0}^{n} k B_{n,k}(x) = nx \ ,$$

and

$$\sum_{k=0}^{n} k(k-1) B_{n,k}(x) = n(n-1)x^2 \ .$$

Proof. The first formula is just the binomial formula which yields $(x+1-x)^n = 1$. The second follows again by the binomial formula by writing

$$\sum_{k=0}^{n} k B_{n,k}(x) = nx \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!((n-1)-(k-1))!} x^{k-1}(1-x)^{(n-1)-(k-1)}$$

and shifting the index $k-1 \to k$. Likewise

$$\sum_{k=0}^{n} k(k-1) B_{n,k}(x) = \sum_{k=0}^{n} n(n-1)x^2 \frac{(n-2)!}{(k-2)!((n-2)-(k-2))!} x^{k-2}(1-x)^{(n-2)-(k-2)}$$

and now shift $k-2 \to k$. The above arguments only work provided that $n \geq 2$. For $n = 1, 2$ they are trivial to verify. □

There is a probabilistic flair to the above formulas that can be expressed in the next lemma.

Lemma 0.2. *We have that*

$$\sum_{k=0}^{n} \frac{k}{n} B_{n,k}(x) = x \ ,$$

and

$$\sum_{k=0}^{n} \left(\frac{k}{n} - x\right)^2 B_{n,k}(x) = \frac{x(1-x)}{n} \ .$$

If we think of the $B_{n,k}(x)$ as probabilities assigned to the 'events' k/n, the first statement says that the expectation value of k/n is x and the second formula says that the variance is $x(1-x)/n$ which is small for n large. The proof is easy and follows from the previous lemma by simple manipulations. Now we are ready to state

Theorem 0.3. *Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Define the polynomial*

$$B_n(f)(x) := \sum f\left(\frac{k}{n}\right) B_{n,k}(x) \ .$$

Then for any $\varepsilon > 0$ there exists N such that for all $n > N$ and all $x \in [0,1]$,

$$|B_n(f)(x) - f(x)| < \varepsilon \ .$$

Note that the polynomial is a Riemann sum but with weights $B_{n,k}(x)$.

Proof. Since f is continuous on the closed interval, it is uniformly continuous, i.e., for every $\varepsilon > 0$ the exists $\delta > 0$ which depends only on ε such that

$$|f(x) - f(y)| < \frac{\varepsilon}{2}$$

whenever $|x - y| < \delta$. We may write, because of Lemma 0.1

$$B_n(f)(x) - f(x) = \sum_{k=0}^{n} [f\left(\frac{k}{n}\right) - f(x)] B_{n,k}(x) \ .$$

Thus,

$$|B_n(f)(x) - f(x)| \le \sum_{|\frac{k}{n}-x|<\delta} |f\left(\frac{k}{n}\right) - f(x)| B_{n,k}(x) + \sum_{|\frac{k}{n}-x|\ge\delta} |f\left(\frac{k}{n}\right) - f(x)| B_{n,k}(x) \ .$$

Note that we have sued the fact that the Bernstein polynomials are non-negative. Hence, we may estimate the right side by

$$|B_n(f)(x) - f(x)| \le \frac{\varepsilon}{2} \sum_{|\frac{k}{n}-x|<\delta} B_{n,k}(x) + 2\max|f(x)| \sum_{|\frac{k}{n}-x|\ge\delta} B_{n,k}(x)$$

In the second term we write

$$\sum_{|\frac{k}{n}-x|\ge\delta} B_{n,k}(x) = \sum_{|\frac{k}{n}-x|\ge\delta} \frac{|\frac{k}{n} - x|^2}{|\frac{k}{n} - x|^2} B_{n,k}(x) \le \frac{1}{\delta^2} \sum_{|\frac{k}{n}-x|\ge\delta} |\frac{k}{n} - x|^2 B_{n,k}(x) \le \frac{x(1-x)}{n\delta^2}$$

Thus, we have

$$|B_n(f)(x) - f(x)| \le \frac{\varepsilon}{2} + 2\max|f(x)| \frac{x(1-x)}{n\delta^2} \le \frac{\varepsilon}{2} + \frac{\max|f(x)|}{2n\delta^2}$$

and choosing

$$n > \frac{\max |f(x)|}{\varepsilon \delta^2}$$

yields the theorem. □