ZORN'S LEMMA

Zorn's lemma is a convenient method of showing that certain mathematical objects exist. It is a highly non-constructive method of proof and does not reveal anything about the mathematical objects under consideration. One could in some sense that is its strength. In essence it is a statement about partially ordered sets.

A partially ordered set is precisely what the name says, a set P with a partial order $\prec$, which has the properties that

$$a)\ a \prec a\ ,\ \text{all } a \in P$$

$$b)\ a \prec b \text{ and } b \prec a \text{ then } a = b$$

$$c)\ \text{If } a \prec b \text{ and } b \prec c \text{ then } a \prec c$$

Note the word partial means that any two elements $a, b \in P$ need not be ordered, i.e., neither $a \prec b$ nor $b \prec a$ necessarily holds.

Example 1: For a good example of a partially ordered set consider all the set P consisting of all subsets of a given set S. For $a, b \in P$ we say that

$$a \prec b \iff a \subset b$$

and we also say that a and b are comparable. Remember that a and b in this example are sets themselves, namely subsets of a given set S. The set P is often denoted as 2^S, the power set of S. Note that two sets a, b for which neither $a \subset b$ nor $b \subset a$ are not comparable. The properties a)-c) are easily verified.

Example 2: Let V be a vector space. Recall that a set of vectors S is linearly independent if and only if any finite linear combination of vectors from S that yields the zero vector is the trivial linear combination, i.e., all coefficients vanish. For ay two sets of linearly independent vectors S_1 and S_2 we say that

$$S_1 \prec S_2$$

if and only if $S_1 \subset S_2$. Once more, this is a partial ordering as the properties a)-c) are easily verified.

A set T is totally ordered if for any $a, b \in T$ one of the following two always holds

$$a \prec b,\ \text{or } b \prec a\ .$$

If P is a partially ordered set then we call any totally ordered collection of elements a **chain**.

Let $C \subset P$ be a totally ordered chain. The element $b \in P$ is an **upper bound** for C if $x \prec b$ for all $x \in C$. Note that we do not assume that b is an element in C.

A **maximal element** in a partially ordered set P is an element $m \in P$ such that

$$m \prec x$$

it follows that $x = m$. The set S in the first example is certainly a maximal element. It is already more difficult to say whether there is a maximal element in the second example and in case it exists what this would be.

Lemma 0.1 (Zorn's lemma)**.** *Let P be a partially ordered set with the property that every chain has an upper bound. Then P has a maximal element.*

Zorn's lemma can be proved using the axiom of choice, in fact it can be shown that it is equivalent. We shall not do so here. A good reference is the book of Halmos 'Naive Set Theory'.

We use Zorn's lemma to show that every vector space has a basis. Recall that the collection of sets consisting of linearly independent vectors forms a partially ordered set P. Let C be a chain. For any two elements $S_1, S_2 \in C$ we have that either $S_1 \prec S_2$ or $S_2 \prec S_1$. For an upper bound we simply consider $S = \cup_{S_\alpha \in C} S_\alpha$. It remains to show that $S \in P$, i.e., that S is linearly independent. Suppose that the set $W = \{v_1, \dots, v_n\} \subset S$ is linearly dependent. For each v_j there exists α_j so that $v_j \in S_{\alpha_j}$. Hence $W \subset \cup_{j=1}^n S_{\alpha_j}$. By changing the enumeration we may assume that $S_{\alpha_j} \subset S_{\alpha_{j+1}}, j = 1, \dots, n-1$ and hence $W \subset S_{\alpha_n}$ which consists of linearly independent vectors. Hence W is linearly independent which is a contradiction. Thus, $S \in P$ and by Zorn's lemma there exists a maximal element B. It remains to show that this is a basis. Suppose not. Then there exists a vector w that is not in the span of B. Hence $B_1 = B \cup \{w\}$ is an element of P and moreover, $B \prec B_1$, however, $B_1 \neq B$. This contradicts the maximality of B. Hence B is a basis for the vector space.