HOMEWORK 10, DUE WEDNESDAY APRIL 24

Problem 5.6.14 (5 points):

Solution: The function e^t is differentiable and its second derivative is positive and hence the function is convex on the whole real line. Jensen's inequality the states that

$$\frac{1}{|E|}\int_E e^f \geq \exp(\frac{1}{|E|}\int_E f)$$

Similarly, $\log x$ is concave and hence, again by Jensen, the result follows, although the left side might be $-\infty$.

Problem 5.6.15 (5 points):

Solution: If ϕ is convex, then it is continuous and the inequality

$$\phi(\frac{x+y}{2}) \leq \frac{\phi(x)+\phi(y)}{2}$$

obviously holds. The converse is more interesting. For any k and non-negative integers α_k, β_k with $\alpha_k + \beta_k = 2^k$ we have that

$$\phi(\frac{\alpha_k x + \beta_k y}{2^k}) \leq \frac{\alpha_k}{2^k}\phi(x) + \frac{\beta_k}{2^k}\phi(y) \ .$$

To see this we first note that by induction for $x_j \in (a,b), j = 1, \dots, 2^k$

$$\phi(\frac{\sum_{j=1}^{2^k} x_j}{2^k}) \leq \frac{\sum_{j=1}^{2^k}\phi(x_j)}{2^k}$$

Now pick $x_j = x, j = 1, \dots, \alpha_k$ and $x_j = y, j = \alpha_k + 1, \dots, 2^k$ and the first statement follows. Pick any $0 < t < 1$. There exists a sequence α_k such that $\alpha_k/2^k \to t$. By the continuity of ϕ we then have that

$$\phi(tx + (1-t)y) \leq t\phi(x) + (1-t)\phi(y) \ .$$

Problem 6.1.21(5 points):

Solution: We assume that $p < q$. Now we write

$$\sum_n |x_n|^q = \sum_n |x_n|^p |x_n|^{q-p} \leq \sup_n |x_n|^{q-p} \sum_n |x_n|^p$$

so that

$$\|x\|_q \leq \|x\|_\infty^{\frac{q-p}{q}} \|x\|_p^{\frac{p}{q}}$$

But trivially

$$\|x\|_\infty \le \|x\|_p$$

and hence

$$\|x\|_q \le \|x\|_\infty^{\frac{q-p}{q}} \|x\|_p^{\frac{p}{q}} \le \|x\|_p^{\frac{q-p}{q}} \|x\|_p^{\frac{p}{q}} = \|x\|_p \ .$$

The sequence given by $\frac{1}{n^r}$ is in ℓ_q but not in ℓ_p for any r with $\frac{1}{p} > r > \frac{1}{q}$.

Problem 6.2.13 (5 points): Write

$$\int_E |f|^p \ge \int_{\{|f|>a\}} |f|^p \ge a^p |\{|f| > a\}| \ .$$

and the inequality follows.

Solution:

Problem 6.2.16 (5 points):

Solution: We assume that $q < p$. Then we write

$$\|f\|_p^p = \int_E |f|^p = \int_E |f|^p \cdot 1 \le [\int_E |f|^{pr}]^{1/r} [\int_E 1^{r'}]^{1/r'}$$

by Hölder's inequality where $\frac{1}{r} + \frac{1}{r'} = 1$. Choose $r = q/p \ge 1$ and hence

$$r' = \frac{r}{r-1} = \frac{q}{q-p}$$

and we get that

$$\|f\|_p^p \le [\int_E |f|^q]^{p/q} |E|^{\frac{q-p}{q}}$$

and hence

$$\|f\|_p \le \|f\|_q |E|^{\frac{1}{p} - \frac{1}{q}}$$

Take the unit ball B in $\mathbb{R}^d$. The function $\frac{1}{|x|^r}$ is in $L^p(B)$ but not in $L^q(B)$ if $\frac{d}{q} < r < \frac{d}{p}$.

b) take the complement of the unit ball, B^c. Then the function $\frac{1}{|x|^r}$ is not in L^p but in L^q for all $\frac{d}{q} > r > \frac{d}{p}$.