HOMEWORK 1, SOLUTIONS

Problem 1, (5 points): Please do problem 0.1.20 in *Introduction to Real Analysis.*
Solution: Let x_n be a Cauchy sequence. By assumption we know that this sequence has a subsequence x_{n_k} which converges to some point x. It remains to show that the whole sequence converges to x. Pick any $\varepsilon > 0$. Since x_n is a Cauchy sequence,there exists N_1 such that for all $n, m > N_1$ the distance $d(x_m, x_n) < \varepsilon/2$. Fix $N > N_1$ such that $d(x, x_{N_1}) < \varepsilon/2$ and note that $d(x, x_n) \le d(x, x_N) + d(x_N, x_n) < \varepsilon/2 + \varepsilon/2 = \varepsilon$ for all $n > N$. This shows that x_n converges to x.

Problem 2, (5 points): Please do problem 0.1.21 in *Introduction to Real Analysis.*
Solution a) We know that $d(x_n, x_{n+1}) < 2^{-n}$. Pick any n, m and we may assume tjat $n > m$. Then

$$d(x_m, x_n) \le d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots d(x_{n-1}, x_n)$$
$$< 2^{-m} + 2^{-m-1} + \cdots + 2^{-(n-1)} < 2^{-m} \sum_{k=0}^{\infty} 2^{-k} = 2^{-m+1} \ .$$

Thus for any $\varepsilon > 0$ we find N so that $2^{-N+1} \le \varepsilon$ and hence for any $n \ge m > N$ we have that $d(x_m, x_n) < \varepsilon$.

b) That x_n is a Cauchy sequence means that for any $\varepsilon > 0$ there exists N so that whenever $n, m > N$, $d(x_m, x_n) < \varepsilon$. Pick $\varepsilon = 1/2$. There exists n_1 so that for all $n > n_1$ $d(x_{n_1}, x_n) < 1/2$. Now pick $\varepsilon = 1/4$. There exists $n_2 > n_1$ so that $d(x_{n_2}, x_n) < 1/4$ for all $n > n_2$. Keep on going this way we find $n_k > n_{k-1}$ so that for all $n > n_k$ we have that $d(x_{n_k}, x_n) < 2^{-k}$. Thus for $k = 1, 2, 3, \dots$ we have that $d(x_{n_k}, x_{n_{k+1}}) < 2^{-k}$.

Problem 3, (7 points): Recall that a metric space X is compact if and only if every open cover has a finite sub-cover. Prove that any sequence in a compact metric space has a convergent sub-sequence.
Solution:
First step: Find a candidate for the limit. Let's start with an arbitrary sequence $x_n, n = 1, 2, 3, \dots$. If the *set* $S = \{x_1, x_2, x_3, \dots\}$ is finite (what is the difference between the set and the sequence?) then the sequence will, after finitely many terms be constant and hence converges. This was the easy case. Now to the more difficult one where S is not finite. Here we have to use compactness. I claim that there exists a point $x \in X$ so that every open ball with x as center contains infinitely many points of the sequence. To see this, assume the contrary, i.e., every point in $y \in X$ is the center of an open ball $B(y)$ that contains only finitely many points of the sequence. The union $\cup_{y \in X} B(y)$ is an open cover of the space X. This space is compact and hence it contains a finite sub-cover, i.e., $X = \cup_{j=1}^{N} B_j$. Each ball contains only finitely many points of the set S and hence the set S must be finite which is a contradiction. Thus, there exists a point x so that every ball centered at x, $B_\varepsilon(x)$ (the open ball of radius ε centered at x), contains infinitely many points of the set S. *It is this point x that is our candidate for the limit.*

Second step: Show that x is indeed the limit for a subsequence. Since every ball centered at x contains infinitely many points of the sequence we may choose n_1 so that $x_{n_1} \in B_1(x)$, then n_2 so that $x_{n_2} \in B_{1/2}(x)$ and so on so that $x_{n_k} \in B_{1/k}(x)$ for all $k = 1, 2, 3, \ldots$. Hence, the sequence x_{n_k} converges to x as $k \to \infty$ and we are done.

Problem 4, (3 points): Prove that every compact metric space is complete.

Solution We have to show that any Cauchy sequence in X converges. Since X is a compact metric space, we know from Problem 3 that every sequence has a convergent subsequences. From Problem 1 we know that any Cauchy sequence, which has a convergent subsequence, converges.

Problem 5, (5 points): Please do problem 0.2.11 in *Introduction to Real Analysis*.

Solution Let $Y \subset X$ be a closed subspace of the complete normed space X and let x_n be a Cauchy sequence in Y. Since x_n is also a Cauchy sequence in X it converges to some element $x \in X$. Since Y is closed, the limit $x \in Y$ and hence Y is complete. Conversely, assume that Y is complete. We have to show that Y is closed. Let $x_n \in Y$ be any sequence that converges in X to some x. Hence, it is a Cauchy sequence in Y and therefore, because Y is complete, it converges in Y to some y. Since limits are unique we have that $x \in Y$. Hence Y is closed.