HOMEWORK 2, SOLUTIONS

Problem 1 (5 points): Please do exercise 0.1.24 (a) in 'Introduction to Real Analysis'.

Solution: The problem, as it is written is not correctly stated. This was pointed out to me by one of your student colleagues. Many thanks!

We prove that the function h is upper semi continuous where it is finite. Note, that no assumptions are made about g.

Assume that x is any point where $h(x) > -\infty$. Let x_n be any sequence in $\mathbb{R}^d$ with limit x. We have to show that

$$\limsup_{n\to\infty} h(x_n) \le h(x) \ .$$

By the definition of the infimum for any $\varepsilon > 0$ there exists $y_0 \in B_r(x)$ such that $g(y_0) - \varepsilon \le h(x) \le g(y_0)$ note that $g(y_0)$ must be finite. We know that, since $B_r(x)$ is open, there exists a ball $B_\varepsilon(y_0) \subset B_r(x)$. Thus, since $x_n \to x$ we must have that the point $y_0 \in B_r(x_n)$ for n sufficiently large. Now,

$$h(x_n) = \inf\{g(y) : y \in B_r(x_n)\} \le g(y_0) \le h(x) + \varepsilon$$

and hence $\limsup_{n\to\infty} h(x_n) \le h(x) + \varepsilon$. Since ε is arbitrary the result follows.

Problem 2 (5 points): Please do exercise 0.1.24 (b) in 'Introduction to Real Analysis'.

Solution: If $x_n \to x$ as $n \to \infty$, then because the function is upper semicontinuous,

$$\limsup f(x_n) \le f(x) \ .$$

By lower semicontinuity,

$$\liminf f(x_n) \ge f(x) \ .$$

This means that $f(x) \le \liminf f(x_n) \le \limsup f(x_n) \le f(x)$ and hence there is equality in these inequalities. Hence $\lim f(x_n) = f(x)$ and f is continuous at x. sequence converges.

Problem 3 (5 points): Please do exercise 1.1.15 in 'Introduction to Real Analysis'.

Solution: a) Let $x \in \cap_{j=1}^\infty \left(\cup_{k=j}^\infty E_k\right)$. This means that $x \in \cup_{k=j}^\infty E_k$ for all $j = 1, 2, \dots$. Thus, for every natural number j there exists a natural number $k \ge j$ such that $x \in E_k$. This means that x cannot belong only to finitely many of the sets E_k. Conversely, if x belongs to infinitely many sets E_k then for any j there exists $k \ge j$ so that $x \in E_k$. Hence, for every j, $x \in \cup_{k=j}^\infty E_k$ and hence $x \in \cap_{j=1}^\infty \left(\cup_{k=j}^\infty E_k\right)$.

b) If $x \in \cup_{j=1}^\infty \left(\cap_{k=j}^\infty E_k\right)$ then there exists some j so that $x \in \cap_{k=j}^\infty E_k$. Thus, there exists some j so that $x \in E_k$ for all $k \ge j$,i.e., $x \in E_k$ in all but finitely many k. Conversely, if $x \in E_k$ for all but finitely many k, then there exists j so that $x \in E_k$ for all $k \ge j$, i.e., there exists j, so that $x \in \cap_{k=j}^\infty E_k$ and hence $x \in \cup_{j=1}^\infty \left(\cap_{k=j}^\infty E_k\right)$.

Problem 4 (5 points): Please do exercise 1.1.16 in 'Introduction to Real Analysis'.

Solution: Assume that $\sum_{k=1}^\infty |E_k|_e < \infty$.

a) By countable sub-additivity

$$|\cup_{k=j}^{\infty} E_k|_e \leq \sum_{k=j}^{\infty} |E_k|_e$$

and since $\cap_{j=1}^{\infty} \left(\cup_{k=j}^{\infty} E_k\right) \in \cup_{k=j}^{\infty} E_k$ for all j, we have that

$$|\cap_{j=1}^{\infty} \left(\cup_{k=j}^{\infty} E_k\right)|_e \leq \sum_{k=j}^{\infty} |E_k|_e$$

for all j but $\lim_{j\to\infty} \sum_{k=j}^{\infty} |E_k|_e = 0$.

b) Since $\cap_{k=j}^{\infty} E_k \in E_k$ for all $k \geq 0$ we have that

$$|\cap_{k=j}^{\infty} E_k|_e \leq |E_k|_e$$

for all $k \geq j$ and since $\sum_{k=1}^{\infty} |E_k| < \infty$ we must have that

$$|\cap_{k=j}^{\infty} E_k|_e = 0 \ .$$

for all $j = 1, 2, \dots$. By countable sub-additivity

$$|\cup_{j=1}^{\infty} \left(\cap_{k=j}^{\infty} E_k\right)|_e \leq \sum_{j=1}^{\infty} |\cap_{k=j}^{\infty} E_k|_e = 0 \ .$$

Problem 5 (5 points): Please do exercise 1.1.20 in 'Introduction to real Analysis'.

Solution: We have to show that for $Q_1, \dots, Q_n$ non-overlapping boxes, we have

$$|\cup_{k=1}^{n} Q_k|_e = \sum_{k=1}^{n} vol(Q_k) \ .$$

One inequality follows from the sub-additivity of the exterior measure, namely

$$|\cup_{k=1}^{n} Q_k|_e \leq \sum_{k=1}^{n} |Q_k|_e = \sum_{k=1}^{n} vol(Q_k)$$

because we proved in the lecture that for a single box Q, $|Q|_e = vol(Q)$. This is Theorem 1.1.7 in 'Introduction to Real Analysis'. The problem is to prove the reverse inequality. Note that the union of finitely many closed boxes is compact. Cover $\cup_{k=1}^{n} Q_k$ by countably many boxes R_j such that $\sum_{j=1}^{\infty} vol(R_j) \leq |\cup_{k=1}^{n} Q_k|_e + \varepsilon$. Enlarge these boxes a tiny bit which yields new open boxes R_j^*, so that R_j is a subset of R_j^* and $vol(R_j^*) \leq (1+\varepsilon) vol(R_j)$. The open boxes cover $\cup_{k=1}^{n} Q_k$ and hence there is a finite sub-cover which we denote by $R_1^*, \dots, R_N^*$. We have that

$$vol(Q_k) = |Q_k|_e \leq \sum_{j=1}^{N} |R_j^* \cap Q_k|_e = \sum_{j=1}^{N} vol(R_j^* \cap Q_k)$$

since the boxes R_j^* cover the Q_ks and the exterior measure of a box and its volume are the same. Hence,

$$\sum_{k=1}^{n} vol(Q_k) = \sum_{k=1}^{n} \sum_{j=1}^{N} vol(R_j^* \cap Q_k) \leq \sum_{j=1}^{N} vol(R_j^*)$$

since for fixed j and $R_j^* \cap Q_k$ are non-overlapping boxes. Hence

$$\sum_{k=1}^{n} vol(Q_k) \leq (1+\varepsilon) \sum_{j=1}^{N} vol(R_j) \leq (1+\varepsilon) \sum_{j=1}^{\infty} vol(R_j) \leq (1+\varepsilon)[| \cup_{k=1}^{n} Q_k|_e + \varepsilon]$$

and since ε is arbitrary, the result follows.