HOMEWORK 3, SOLUTIONS

Problem 1 (5 points): Please do problem 1.1.31 in 'Introduction to Real Analysis'.

Solution: Subdivide the real line into intervals $[k, k+1]$. The function f is continuous on the real line and hence uniformly continuous on the interval $[k, k+1]$. Pick any ε. There exists $\delta > 0$ so that $|f(x) - f(y)| < \varepsilon$ for $x, y \in [k, k+1]$ with $|x - y| < \delta$. Subdivide the interval $[k, k+1]$ into n subintervals so that $1/n < \delta$. Denote these intervals by $I_1, \dots, I_n$ and by x_j the mid point of the interval I_j. Consider the closed box Q_j centered at the point $(x_j, f(x_j))$ that has length $1/n$ along the x-axis and length 2ε in the direction of the y axis. By construction, these boxes cover the graph of the function over the interval $[k, k+1]$. Each has volume $2\varepsilon/n$ and since there are n boxes, the total area is 2ε. Since ε can be chosen arbitrarily small, we find that the graph of f over the interval $[k, k+1]$ has two dimensional exterior measure zero. The whole graph is a countable union of sets of exterior measure zero and hence, by countable subadditivity, has measure zero.

Problem 2 (5 points): Please do problem 1.1.35 in 'Introduction to Real Analysis'.

solution: That the space $\mathbb{R}^{d-1} \times \{0\}$ has measure zero as a subset of $\mathbb{R}^d$ can be seen by considering the closed unit cubes C_p in $\mathbb{R}^{d-1}$ that are centered at each point $p \in \mathbb{Z}^{d-1}$. They are a countable number. Now the measure of each unit cube is zero. Just take the box

$$C_p \times [-\varepsilon, \varepsilon]$$

which has exterior measure 2ε and since ε is arbitrary it follows that $|C_p| = 0$. the rest follows by countable subadditivity.

Problem 3 (5 points): Please do problem 1.1.38 in 'Introduction to Real Analysis'.

Solution: Suppose there exist countable many boxes with $\sum_k vol(Q_k) < \infty$ and each point of E belongs to infinitely many boxes. This means that

$$E \subset \cap_{j=1}^{\infty}(\cup_{k=j}^{\infty} Q_k)$$

and hence

$$|E|_e \leq |\cup_{k=j}^{\infty} Q_k|_e$$

for all $j \geq 1$. Because, $\sum_{k=1}^{\infty} vol(Q_k) < \infty$

$$|E|_e \leq \sum_{k=j}^{\infty} vol(Q_k) \to 0$$

as $j \to \infty$. Conversely, assume that $|E_e = 0$. For any n there exist countably many boxes Q_n^k so that $E \subset \cup_{k=1}^{\infty} Q_n^k$ and such that

$$\sum_{k=1}^{\infty} vol(Q_n^k) < 2^{-n} \ .$$

Pick any $x \in E$, then $x \in \cup_{k=1}^{\infty} Q_n^k$ for all n and hence for any n there must be a $k(n)$ such that $x \in Q_n^k$. It could be that all but finitely many of these boxes are the same. However that cannot be, because if the were infinite repetitions of the same box the sumof the volumes could not be finite. Hence x belongs to infinitely may boxes.

Problem 4 (5 points): Please do problem 1.2.31 in 'Introduction to Real Analysis'.

Solution: The distance between the set F and the set K is given by

$$d(F,K) := \inf_{x\in F, y\in K} \|x-y\| \ .$$

Pick any $y_0 \in K$ and consider a closed ball $\overline{B}_R(y_0)$ of radius R centered at y_0 such that $K \subset \overline{B}_R(y_0)$ and $F \cap \overline{B}_R(y_0) \neq \emptyset$. Such a ball exists, because K is compact and if the there is no intersection with F for any R then F must be the empty set contrary to our assumption. By the triangle inequality, any point in F outside $\overline{B}_R(y_0)$ has distance larger than $d(F,K)$. Hence it follows that

$$d(F,K) = \inf_{x\in F\cap\overline{B}_R(y_0), y\in K} \|x-y\| \ .$$

The set $F\cap\overline{B}_R(y_0)$ is compact because closed and bounded sets in $\mathbb{R}^d$ are compact and hence $F\cap\overline{B}_R(y_0)\times K$ is a compact set in $\mathbb{R}^d\times\mathbb{R}^d$. The function $(x,y)\to\|x-y\|$ is continuous as a function of two variables and hence has a minimum on $F\cap\overline{B}_R(y_0)\times K$. Denote the points of minimal distance by (x_m, y_m). Because F and K are disjoint $x_m \neq y_m$ and hence $d(F,K) > 0$.

Problem 5 (5 points): Please do problem 1.2.32 in 'Introduction to Real Analysis'.

Solution: We write the sets as disjoint unions. Recall that if A, B are measurable so is $A\cap B$ and $A\setminus(A\cap B) = A\cap(A\cap B)^c$. Now we write that various sets as disjoint unions of measurable sets:

$$A = [A\cap(A\cap B)^c]\cap(A\cap B) \ , B = [B\cap(A\cap B)^c]\cap(A\cap B)$$

and

$$A\cup B = [A\cap(A\cap B)^c]\cup(A\cap B)\cup[B\cap(A\cap B)^c] \ .$$

Since the sets are disjoint and measurable

$$|A|+|B| = |A\cap(A\cap B)^c| + 2|A\cap B| + |B\cap(A\cap B)^c|$$

and

$$|A\cup B| = |A\cap(A\cap B)^c| + |A\cap B| + |B\cap(A\cap B)^c|$$

Hence it follows that

$$|A|+|B| = |A\cap(A\cap B)^c| + 2|A\cap B| + |B\cap(A\cap B)^c| = |A\cup B| + |A\cap B| \ .$$